# Spaces in Geometric Representation Theory

## Lecture 1:

**Organisation:**

- 2 lectures + 1 problem session / week

  ↑ Present at least one problem at the blackboard!

- Oral exam (date TBD)

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Content:

0. Linear algebraic groups: Rapid overview.

1. Reductive groups and flag varieties

2. Kac-Moody groups —— " ——

3. The horocycle correspondence

Prerequisits:

- semisimple Lie groups/algebras

- basic AG

Sources:

- Classification des groupes Lie algébriques [Bible]
- Schémas en groupes [SGA 3]
- Linear Algebraic Groups : Borel, Humphreys
- ... see website

# 0 Linear algebraic groups: Rapid overview.

⚠ Try to avoid scheme-theoretic technicalities & non-split and char $p$ phenomena

Global assumptions:

$k$ alg. closed field $\operatorname{char} k = 0$

$\mathrm{Var}/k$ = reduced, separated, finite type schemes$/k$

$\cancel{\operatorname{Spec} k[x]/x^2}$ $\quad$ $\cancel{\mathbb{A}^1 \sqcup_{\mathbb{G}_m} \mathbb{A}^1}$ $\quad$ $\cancel{\operatorname{Spec} k[\underbrace{x_1, x_2, \ldots}_{\infty}]}$

(Quasi)projective variety$/k$ = (locally) closed in $\mathbb{P}^n/k$

open $\cap$ closed

Write $\mathcal{O}(X) = \mathcal{O}_X(X)$ for regular functions.

## 0.1. Affine algebraic groups

Def.: An (affine) algebraic group $\mathcal{G}$ is a group object in the category of (affine) varieties.

Thm: There is an equivalence of categories

$$\{\text{affine alg. grp's}/k\} \longleftrightarrow \{\text{commutative, reduced, f.g Hopf alg.}/k\}$$

$$\mathcal{G} \longmapsto \mathcal{O}(\mathcal{G})$$

$$\operatorname{Spec} A \longleftarrow A$$

$\square$

Rmk: Under this equivalence we have, for example,

$$\mathcal{G} \text{ commutative} \iff \mathcal{O}(\mathcal{G}) \text{ cocommutative}$$

$$V \text{ representation of } \mathcal{G} \iff V \text{ comodule for } \mathcal{O}(\mathcal{G}) \quad \square$$

**Example**: Let $GL_n = \mathrm{Spec}(k[x_{ij}, d]/\det(x_{ij})d = 1) \subset \mathbb{A}^{n^2+1}$

Then $GL_n$ is an affine algebraic group, with Hopf algebra with comultiplication, antipode and counit

$$\mu^*: \mathcal{O}(GL_n) \to \mathcal{O}(GL_n) \otimes \mathcal{O}(GL_n), \quad x_{ij} \mapsto \sum x_{ik} \otimes x_{kj}$$

$$\iota^*: \mathcal{O}(GL_n) \to \mathcal{O}(GL_n), \quad x_{ij} \mapsto (-1)^{i+j} \det(x_{i'j'})_{\substack{i' \neq j \\ j' \neq i}} d$$

$$e^*: \mathcal{O}(GL_n) \to k, \quad x_{ij} \mapsto \delta_{ij} \qquad \square$$

**Exercise**: Work out the Hopf algebra for $G = \mathbb{G}_m$ and $\mathbb{G}_a$ $\square$

**Lemma**: Let $G$ be an algebraic group. Then $G$ is smooth. Let $G^\circ$ be the irreducible component of $e$. Then $G^\circ \trianglelefteq G$, $\pi_0(G) = G/G^\circ$ is f.t. and $G = \bigoplus_{g \in \pi_0(G)} g G^\circ$ $\square$

## 0.2. Linearization

**Def:** A linear algebraic group is a closed subgroup of $GL_n$

A linear alg. grp. is affine. We now show the converse.

**Lemma:** Let $G$ be an aff. alg. group acting on an affine variety $X$ via $\varphi: G \times X \to X$. Then

(1) for each $f \in \mathcal{O}(X)$, there is a f.d. $G$-stable subspace such that $f \in V \subset \mathcal{O}(X)$

(2) $X$ admits a $G$-eq. closed immersion $X \hookrightarrow V$ into a f.d. $G$-rep. $V$

<u>Proof</u>: (1) Write $\varphi(f) = \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{O}(G) \otimes \mathcal{O}(X)$.

Then, for $x \in G$, $(gf)(x) = \sum_{i=1}^{n} a_i(g^{-1}) \otimes b_i(x)$ and

$V = \langle f_1, \dots, f_n \rangle$ does the job.

(2) Choose generators $f_1, \dots, f_n$ of $\mathcal{O}(X)$. Via (1),

there is a f.d. $G$-stable $V \subset \mathcal{O}(X)$, s.t.

$G f_i \subset V$. Since $V$ generates $\mathcal{O}(X)$ we obtain

a $G$-linear map $\operatorname{Sym} V = \mathcal{O}(V^*) \twoheadrightarrow \mathcal{O}(X)$ which

induces a closed immersion $X \to V^*$ $\square$

Prop: Let $G$ be an alg. grp. acting on a variety $X$.

Let $x \in X$. Then

(1) $G \cdot x \subset X$ is locally closed

(2) $\dim G \cdot x = \dim G - \dim G_x$

(3) $\overline{G \cdot x}$ is a union of orbits of smaller dimension.

Pf Omitted. Uses constructibility of $G \cdot x$. $\square$

Thm: (1) Let $f : G \to H$ be a hom. of alg. grp's.

Then $\operatorname{im} f \subset H$ is closed. If $\ker f = \{e\}$, then $f$

is a closed immersion.

(2) Let $G$ be an affine alg. group. Then $G$ is linear.

Pf: (1) $\mathrm{im}(f)$ closed follows from Prop., closed immersion follows from Zariski main thm.

(2) Use Lemma for $X = \mathcal{G}$ and (1) $\square$

## Lecture 2:

<u>Addendum:</u> $\mathcal{G}$ alg. grp. acting on variety $X$ via $\varphi: \mathcal{G} \times X \to X$. Then $\mathcal{G}$ acts $k$-linearly on $\mathcal{O}(X)$ via $(gf)(x) = f(g^{-1}x)$, for $g \in \mathcal{G}$, $x \in X$, $f \in \mathcal{O}(X)$.

The action map $\varphi$ yields

$$\varphi^*: \mathcal{O}(X) \to \mathcal{O}(\mathcal{G} \times X)$$

$$f \mapsto ((g,x) \mapsto f(g \cdot x))$$

Now we identify

$$\mathcal{O}(\mathcal{G} \times X) \cong \mathcal{O}(\mathcal{G}) \otimes \mathcal{O}(X)$$

$$(g,x) \mapsto a(g)b(x) \longleftarrow a \otimes b \, .$$

So if $\varphi^*(f) = \sum_{i=1}^{n} a_i \otimes b_i$, then

$$\varphi^*(f)(g, x) = \sum_{i=1}^{n} a_i(g)\, b_i(x)$$

and $(g \cdot f)(x) = f(g^{-1}x) = (\varphi^*(f))(g^{-1}, x) = \sum a_i(g^{-1})\, b_i(x)$

## 0.3. Quotients

Lem: (Chevalley) Let $G$ be lin. alg. $H < G$ closed subgroup. Then there is a $G$-rep. $V$ and f.dim. subspace $L \subset V$, s.t., $G_L = \{g \in G \mid gL = L\} = H$

Proof: We have $H = \{g \in G \mid gH = H\}$.

Let $0 \to I(H) \to \mathcal{O}(G) \to \mathcal{O}(H) \to 0$. Then $H = G_{I(H)}$

Choose f.d. subspaces $U \subset I(H)$ and $U \subset W \subset \mathcal{O}(G)$, s.t., $\mathcal{O}(G)U = I(H)$, $HU = U$, $GW = W$. Then $H = G_U$. Now, pass to $L = \Lambda^{\dim U} U \subset \Lambda^{\dim U} W = V$.

Exercise: $H = G_L$ $\square$

<u>Def</u>: Let $G$ be an alg. grp., $X$ a $G$-variety. A $G$-morphism $\pi: X \to Y$ is called a geometric quotient if

(1) $\pi$ is a set-theoretical quotient $X \to X/G$

(2) $\pi$ is open (3) $(\pi_* \mathcal{O}_X)^H = \mathcal{O}_Y$ $\square$

<u>Thm</u>: Let $H < G$ be closed subgroup of an affine alg. grp. Then there is a geometric quotient $\pi: G \to G/H$, s.t. $G/H$ is smooth and quasi-projective.

If $H$ is normal in $G$, then $G/H$ is affine

<u>Pf</u>: By Chevalley's Thm., there is a $G$-rep. $V$, s.t. $H = G_x$ for $x = [L] \in \mathbb{P}(V)$. Obtain a map:

$\pi : G \to X = G\cdot x \subset \mathbb{P}(V)$. The map is smooth (Why?)

For (3), consider

$$\begin{tikzcd}
G \times H \arrow[dr, "f"] \arrow[drr, bend left, "\mu"] \arrow[ddr, bend right, "p"'] & & \\
& G \times_{G/H} G \arrow[r] \arrow[d] & G \arrow[d, "\pi"] \\
& G \arrow[r, "\pi"'] & G/H
\end{tikzcd}$$

Now $f$ is bij and by ZMT an iso since $G \times_{G/H} G$ is smooth. Now by smooth base change:

$$(\pi^* \pi_* \mathcal{O}_G)^H \cong (p_* \mu^* \mathcal{O}_G)^H = (\mathcal{O}_G \otimes \mathcal{O}(H))^H = \mathcal{O}_G$$

Since $\pi$ is smooth, it is faithfully flat, and

$(p_* \mathcal{O}_G)^H = \mathcal{O}_X$ $\square$

## 0.4. Lie algebras

<u>Definition</u>: Let $\mathcal{G}$ be an affine algebraic group

Then the Lie-algebra of $\mathcal{G}$ is

$$\mathfrak{g} = \mathrm{Lie}\,\mathcal{G} = T_e\mathcal{G} = \mathrm{Der}(\mathcal{O}_{\mathcal{G},e}) \cong \mathrm{Der}_{\mathcal{G}}(\mathcal{O}_{\mathcal{G}})$$

$$\mathrm{Der}_{\mathcal{G}}(\mathcal{O}_{\mathcal{G}}) = \{\delta \in \mathrm{Der}\,\mathcal{O}_{\mathcal{G}} \mid g\delta = \delta\}$$

Carry a Lie bracket
$[\delta, \delta'] = \delta\delta' - \delta'\delta$

This yield a functor

$$\begin{aligned} \mathrm{Lie} : \{\text{affine alg. grp.}\} &\longrightarrow \{\text{Lie algebras}\} \\ \mathcal{G} &\longmapsto \mathrm{Lie}(\mathcal{G}) \\ f &\longmapsto df \end{aligned}$$

$\square$

Example: $\mathfrak{gl}_n = \mathrm{Lie}(GL_n) = T_e GL_n = T_e M^{n\times n} = M^{n\times n}$

with the standard Lie bracket. $\square$

Exercise (1) Compute the differential of multiplication and inversion.

(2) Compute $\mathfrak{so}_n = \mathrm{Lie}\, SO_n \subset \mathfrak{gl}_n = GL_n$ in nice ways $\square$

There are two adjoint representations

$$\mathrm{Ad}: G \to GL(\mathfrak{g}), \quad \mathrm{ad} = d\mathrm{Ad} \quad \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

For $G \subset GL_n$, $\mathfrak{g} \subset \mathfrak{gl}_n$, we get

$$\mathrm{Ad}(g)X = gXg^{-1}, \quad \mathrm{ad}(X)Y = [X, Y].$$

Thm: (1) Let $G$ be a connected affine alg. grp. Then

$$\mathrm{Lie} : \{ H \subset G \mid H \text{ closed connected} \} \to \{ \mathfrak{h} \subset \mathfrak{g} \}$$

is injective.

(2) Let $H \subset GL(V)$ be a closed subgroup, $W \subset V$, $v \in V$.

Then $\mathrm{Lie}(H_W) = \mathfrak{h} \cap \mathfrak{g}_W$, $\mathcal{L}(H_v) = \mathfrak{h} \cap \mathfrak{g}_v$ $\square$

Pf: Omitted. Uses char $k = 0$ $\square$

Cor: Let $G$ be a connected affine algebraic group and $V$ a $G$-rep. Then $W \subset V$ is a $G$-subrep. if and only if it is a $\mathfrak{g}$-subrep.

## 0.5 Jordan-Chevalley

Let $X \in \mathfrak{gl}_n$. Then, the Jordan decomposition yields unique $X_n, X_s \in \mathfrak{gl}_n$, s.t.:

(1) $X_n + X_s = X$ (2) $X_n X_s = X_s X_n$

(3) $X_s$ is diagonalizable (4) $X_n$ is nilpotent

(5) $X_s = p(X)$, $X_n = q(X)$, for polynomials $p, q$.

Similarly, for $x \in GL_n$, we obtain $x_u = 1 + x_s^{-1} x_n$

where $x_u$ is unipotent and $x_s x_u = x_u x_s = x$

If $x \in GL_n$ is unipotent we obtain a closed map

$$\mathbb{G}_a \hookrightarrow GL_n, \quad a \mapsto \exp(an)$$

Thm (Jordan-Chevalley): Let $\mathcal{G}$ be an affine alg. grp.

Then $x \in \mathcal{G}$, $X \in \mathfrak{g}$ admit intrinsic Jordan

decompositions $x = x_s x_u$, $X = X_s + X_n$ with $x_s, x_u \in \mathcal{G}$,

$X_s, X_u \in \mathfrak{g}$, compatible with homomorphism $\mathcal{G} \to H$

Pf: Omitted $\square$

Exercise: Show that semisimple/unipotent elements

in $SL_2$ do not form a subgroup.

## 0.6. Diagonalizable groups

Def: An affine alg. group $D$ is called diagonalizable if $D \hookrightarrow (\mathbb{G}_m)^n$ or $D$ is commutative and $d = d_s$ $\forall d \in D$

If $D \cong (\mathbb{G}_m)^n$ is called a torus $\square$

Def: For an algebraic group $\mathcal{G}$, we define

$$X(\mathcal{G}) = \mathrm{Hom}_{grp}(\mathcal{G}, \mathbb{G}_m), \quad Y(\mathcal{G}) = \mathrm{Hom}_{grp}(\mathbb{G}_m, \mathcal{G})$$

the character and cocharacter lattice. There is a pairing $\langle , \rangle : X(\mathcal{G}) \times Y(\mathcal{G}) \to \mathrm{Hom}_{grp}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$X(\mathcal{G})$ is a group. If $\mathcal{G}$ is commutative, $Y(\mathcal{G})$ is a group and $\langle , \rangle$ bilinear $\square$

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Exercise: (1) Prove that $\mathrm{Hom}_{grp}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

(2) Let $T = (\mathbb{G}_m)^n$. Compute $X(T)$, $Y(T)$ and $\langle , \rangle$

(3) Let $G = \mu_n = \mathrm{Spec}(k[x]/x^n - 1)$. Show that $\mu_n$ is diagonalizable. Compute $X(\mu_n)$, $Y(\mu_n)$ and $\langle , \rangle$ $\square$

Thm: There is an equivalence of categories

$$\{\text{diagonalizable grp's}\} \longrightarrow \{\text{f.g. abelian grp's}\}^{op}$$

$$D \longmapsto X(D)$$

$$\mathrm{Spec}(k[X]) \longleftarrow\!\shortmid X$$

$\square$

## 0.7. Solvable groups

<u>Def</u>: A group $G$ is called solvable, if its derived series

$$G \supset [G,G] \supset [[G,G],[G,G]] \supset \dots$$

terminates in $\{e\}$ in f.t. many steps

<u>Example</u>: (1) Commutative $\Rightarrow$ solvable

(2) Subgroups and quotients of $G$ are solvable if $G$ is

(3) $B_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset GL_n$ is solvable (<u>Exercise</u>)

<u>Thm</u> (Lie-Kolchin+) (1) Let $\mathcal{G}$ be connected and solvable acting on a complete variety $X$. Then $X^{\mathcal{G}} = \{x \in X \mid gx = x \ \forall g \in \mathcal{G}\} \neq \emptyset$

(2) A connected solvable aff. alg. grp. $\mathcal{G}$ is of the form

$$\mathcal{G} = \underset{\substack{\uparrow \\ \text{max. torus}}}{T} \ltimes \underset{\substack{\nwarrow \\ \text{unipotent elements in } \mathcal{G}}}{\mathcal{G}_u}$$

$T$ is unique up to conjugation $\square$

## 0.8. Aside: Plücker Coordinates

Idea: Express subspace relations in a vector space $V$ via $\mathbb{P}(\Lambda^\bullet V)$. $\square$

Def: Let $V$ be an $n$-dim. vector space. The (set-theoretical) Grassmannian of $d$-dim. subspaces in $V$ is

$$Gr_d(V) = \{ W \subset V \mid \dim W = d \} \qquad \square$$

**Prop:** Let $V$ be a $n$-dim vector space.

Then, the map, called Plücker embedding,

$$i: \; Gr_d(V) \to \mathbb{P}(\Lambda^d V)$$

$$W = \langle w_1, \dots, w_d \rangle \longmapsto w_1 \wedge \dots \wedge w_d$$

is well-defined and injective.

**Proof:** Let $g \in GL(W)$, then

$$gw_1 \wedge \dots \wedge gw_d = (\det g)\, w_1 \wedge \dots \wedge w_d$$

and hence both define the same element in $\mathbb{P}(\Lambda^k(V))$.

Injectivity follows from the next lemma. $\square$

Call a vector $0 \neq v \in \Lambda^d V$ is called simple if there are $v_i \in V$, s.t.

$v = v_1 \wedge \dots \wedge v_d$.

<u>Lemma</u>: Let $0 \neq v \in \Lambda^d V$, $W < V$. Then

$$v \wedge W = \{0\} \iff v \in \Lambda^d W, \ \dim W = d$$

In particular, in this case $v$ is simple (if $v \neq 0$).

<u>Proof</u>: <u>Exercise</u>.

<u>Thm</u>: (Plücker relations) Let $0 \neq v \in \Lambda^d V$.

$W = \operatorname{im}(\Lambda^{d-1} V^* \to V, \ \alpha \mapsto \iota_\alpha(v))$. Then

$$v \text{ simple} \iff v \wedge W = \{0\}$$

<u>Proof</u>: <u>"$\Rightarrow$"</u> Easy exercise <u>"$\Leftarrow$"</u> Previous lemma.

Exercise: Choosing a basis of $V$, write the condition

$$v \wedge w = 0$$

as a set of homogeneous quadratic relations in $\Lambda^d V$. These are the classical Plücker relations.

Corollary $Gr_k(V) \subset \mathbb{P}(\Lambda^k V)$ is a projective variety.

Remark: Along these lines one may also show that for any sequence $\underline{d} = 0 < d_1 < \cdots < d_k < d$ the flag variety $\mathcal{F}\ell_{\underline{d}}(V) = \{V_1 \subset \cdots \subset V_k \subset V \mid \dim V_i = d_i\}$ is projective

Lecture 4

# 1. Reductive groups and flag varieties

## 1.1. Borel subgroups and maximal tori

Def: Let $G$ be an aff. alg. grp.

A Borel subgroup (maximal torus) of $G$ is a maximal connected (semisimple) solvable subgroup of $G$. $\square$

Thm: Let $B \subset G$ be a Borel subgroup.
All Borel subgroups of $G$ are conjugate to $B$ and $G/B$ is a projective variety.

Let $H \subset \mathcal{G}$ be a Borel subgroup of maximal dimension.

Let $L \subset V$ be as in 0.3. Lemma. Then $H$ acts

on $\mathcal{F}\ell(V/L)$, which is projective (0.8 Remark).

Hence there is a fixed point $F' = (V_2/L \subset \dots \subset V_n/L)$, which

yields a flag $F = (V_1 = L \subset V_2 \ \dots \ \subset V_n) \in \mathcal{F}\ell(V)$, s.t.

$\mathcal{G}_F = H$ and $\mathcal{G}/H \to \mathcal{G}.F$ is bij.

Since the stabilizer of any flag is solvable and $H$ maximal,

$\mathcal{G}.F$ has minimal dimension and is hence closed.

So $\mathcal{G}/H$ is projective (This needs an AG argument).

Now $B$ acts on $\mathcal{G}/H$ and has a fixed point, say $gH$.

Hence $BgH = gH \Leftrightarrow gHg^{-1} \subset B$. This is an equality since both are Borel subgroups $\square$

Moreover we have the following facts, some closely related to the Theorem, which we will not prove here:

Facts: (1) For $P$ parabolic $N_G(P) = P$

(Normalizer Thm.)

(2) If $T \subset B \subset G$ is a maximal torus in a Borel subgroup,

then $C = C_G(T) \subset B$ and $N_B(T) = C_B(T)$

Moreover $C = T \times C_u$ and $\bigcup_{g \in G} g^{-1} C g \subset G$ is dense

(3) If $D \subset G$ is diagonalizable, then $N_G^{\circ}(D) = C_G^{\circ}(D)$

(Rigidity of tori)

(4) If $H \subset G$ and $D \subset N_G(H)$ normalizes $H$, then

$\mathrm{Lie}(C_G(H)) = \mathfrak{c}_{\mathfrak{g}}(H) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(h)(X) = X \ \forall h \in H\}$

(Infinitesimal = global centralizer)

<u>Cor</u>: (1) All maximal tori $T$ in $G$ are conjugate

Call $\operatorname{rank} G = \dim T$

(2) $G/P$ projective $\Leftrightarrow$ $P$ contains a Borel

Call $P$ <u>parabolic</u> $\square$

(3) The following map is an isomorphism

$$G/B \longrightarrow \mathcal{B} = \{B' \subset G \mid B' \text{ Borel subgroup}\}$$

$$gB \longmapsto gBg^{-1}$$

Call $\mathcal{B}$ the full <u>flag variety</u> associated to $G$.

(4) If $\varphi: G \to G'$ is surjective, then it preserves

maximal tori, Borel subgroups, parabolics.

Example: Consider $B_n = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset GL_n$. Then $GL_n / B_n \xrightarrow{\sim} \mathcal{F}\ell(k^n)$ is projective, so $B_n$ is parabolic and solvable. Hence $B_n$ is a Borel subgroup.

## 1.2. Weyl group – Basics

Def: For $H \subset G$, $W(G,H) = N_G(H)/C_G(H)$ is called the Weyl group of $H$ in $G$. $\square$

Clearly, $W(G,H)$ only depends on the conjugacy class of $H$ in $G$. For an affine alg. group $G$, the Weyl group is $W = W(G) = W(G,T)$ for a maximal torus $T \subset G$.

Rem.: By Fact 1.1. (3) $W(G,H)$ is finite if $H$ is any diagonalizable subgroup of $G$. So $W = W(G)$ is finite.

**Proposition:** Let $T$ be a maximal torus. Then

Then $\mathcal{B}^T$ is a $W(G,T)$-torsor.

**Proof:** Note that $T \in \mathcal{B}^T \Leftrightarrow T \subset N_G(B) = B$, where we used Fact 1.1.(1). By Fact 1.1.(2) $C_G(T) \subset B$ for if $T \subset B$. Hence, the action of $W(G,T)$ on $\mathcal{B}^T$ is well-def..

**Transitivity:** Let $B, B' \in \mathcal{B}^T$. Then there is $x \in G$ with $B' = xBx^{-1}$ (Thm. 1.1). Then $T, xTx^{-1} \subset B'$ are maximal tori, so there is $y \in B'$ with $T = yxTx^{-1}y^{-1}$. Hence $z = yx \in N_G(T)$ and $zBz^{-1} = B'$

**Freeness:** Let $x \in N_G(T)$, $B \in \mathcal{B}^T$, s.t. $xBx^{-1} = B$.

So $x \in N_G(B) = B$. Hence $x \in B \cap N_G(T) = N_B(T) = C_G(T)$

where we used Fact 1.1.(2). So, $e = [x] \in W(G,T)$ $\square$

In particular, $\mathcal{B}^T$ is finite.

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<u>Fact</u>: Let $\varphi: G \to G'$ be surjective, $T \subset G$, $T' = \varphi(T) \subset G'$

max. tori. Then $\mathcal{B}^T \to \mathcal{B}'^{T'}$ and $W(G,T) \to W(G',T')$

are surj.. Moreover, if $\ker\varphi \subset \bigcap_{B \in \mathcal{B}} B$, then these

maps are isomorphisms $\square$

# 1.3. Summoning Fixed Points

**Lemma:** Let $\mathbb{G}_m$ act on $k^n$ via $\lambda (a_i) = (\lambda^{m_i} a_i)$.

and $v = (a_i) \in k^n$. Denote

$$I = \{ i \in I \mid a_i \neq 0 \}, \quad m_{min} = \min_{i \in I} m_i, \quad m_{max} = \max_{i \in I} m_i$$

$$I_{min} = \{ i \in I \mid m_i = m_{min} \}, \quad I_{max} = \ldots$$

$$v_{min} = \sum_{i \in I_{min}} a_i e_i \qquad v_{max} = \ldots$$

Then, the action of $\mathbb{G}_m$ on $[v] \in \mathbb{P}(k^n)$ can be extended

$$\begin{tikzcd}
\lambda \arrow[r, mapsto] & {[\lambda v]} \\
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{P}(k^n) \\
\mathbb{P}^1 \arrow[ur] &
\end{tikzcd}$$

via $\quad 0 \mapsto [v_{min}], \quad \infty \mapsto [v_{max}]$

Moreover $[v_{min}]$ and $[v_{max}]$ are fixed points which are distinct if $[v]$ is not a fixed point.

Proof: We have

$$\lambda[a_i] = [\lambda^{m_i} a_i] = [\lambda^{m_i - m_{min}} a_i] = [\lambda^{m_i - m_{max}} a_i]$$

So, since $m_i - m_{min} \geq 0$ and $m_i - m_{max} \leq 0$, the expression is well-defined for $\lambda = 0, \infty$, and yields $[v_{min}]$ and $[v_{max}]$, respectively.

**Thm**: In the notation of the lemma, let $X \subset \mathbb{P}(k^n)$ be an irreducible closed subset.

(a) If $\dim X > 1$, $|X^{\mathbb{G}_m}| \geq 2$

(b) If $\dim X > 2$, and $X \not\subset \mathbb{P}(W)$ for all $W \subset k^n$ with $\dim W = n-1$, $|X^{\mathbb{G}_m}| \geq 3$

**Proof**: (a) Since $\dim X > 1$, $|X| > \infty$. If $X^{\mathbb{G}_m} = \mathbb{G}_m$ we are done. Else, let $v \in X - X^{\mathbb{G}_m}$. We obtain, as in lemma,

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "(*)"] \arrow[d] & X \arrow[r] & \mathbb{P}(k^n) \\ \mathbb{P}^1 \arrow[ur, "(**)"] \arrow[urr] & & \end{tikzcd}$$

where $(*)$ exists since $X$ is $\mathbb{G}_m$-stable and $(**)$ since $X$ is closed. $[v_{min}], [v_{max}] \in X$ are the desired two fixed points.

(b) Proceed as in (a), assume that $[v] \in X - X^{\mathbb{G}_m}$. Choose

$i^* \in I_{max}$ and let $W = \langle e_i \mid i \neq i^* \rangle$.

Then by general AG fact, irreducible components of

$X \cap \mathbb{P}(W)$ have codim. 1 in $X$. Choose one of them,

say $Y$. By induction $|Y^{\mathbb{G}_m}| = 2$.

Now $[v_{max}] \in X - \mathbb{P}(W)$ is the third fixed point $\square$

<u>Corollary</u>: (1) Let $P \subsetneq G$ be a parabolic subgroup in

a lin. alg. grp., $T \subset G$ any torus. Then $|(G/P)^T| \geq 2$

(or $\geq 3$ if $\dim G/P > 1$)

(2) If $\dim \mathcal{B} \geq 1$, then $|\mathcal{B}^T| = |W| \geq 2$.

(3) If $T \subset G$ is a maximal torus, then $G$ is generated by $\mathcal{B}^T$

<u>Proof</u>: (1) Sketch: Choose $G$-rep. $V$, such that $G/P \xrightarrow{\sim} X \subset \mathbb{P}(V)$ equivariantly. Wlog. $X \not\subset \mathbb{P}(W)$ for any $W \not\subset V$. Now, choose cocharacter $\lambda: \mathbb{G}_m \to T$, such that $X^T = X^{\mathbb{G}_m}$. Then apply Theorem

(2) Clear (3) Omitted $\square$

## 1.3. Reductivity / Semisimplicity

<u>Def + Lemma:</u> Let $G$ be an connected aff. alg grp. Then $G$ contains a unique largest normal connected (unipotent) solvable subgroups

$$R_u(G),\, R(G) \trianglelefteq G$$

called the (unipotent) radical of $G$.

$G$ is called reductive or semisimple if $R_u(G)$ or $R(G)$ are trivial, resp.

<u>Sketch:</u>

$$\overbrace{R_u(G) \subseteq R(G^\circ)}^{\text{torus}} \subseteq \overbrace{G^\circ \subseteq G}^{\text{finite}}$$

$$\underbrace{R_u(G) \subseteq \underbrace{R(G^\circ) \subseteq G^\circ}_{\text{semisimple}}}_{\text{reductive}} \subseteq G$$

Remark: (1) One may show that:

$$\mathcal{R}(\mathcal{G}) = \bigcap_{B \in \mathcal{B}} B \quad \text{and} \quad \mathcal{R}_u(\mathcal{G}) = \bigcap_{B \in \mathcal{B}} B_u .$$

(2) The semi-simple (reductive) rank of $\mathcal{G}$ is

$$\operatorname{rank}_{ss} \mathcal{G} = \operatorname{rank} \mathcal{G}/\mathcal{R}(\mathcal{G}) \text{ and } \operatorname{rank}_{red} \mathcal{G}/\mathcal{R}_u(\mathcal{G}) \quad \square$$

Thm: Let $\mathcal{G}$ be a connected lin. alg. grp., $T \subset \mathcal{G}$ max. torus and $W = W(\mathcal{G}, T)$. Then t.f.s.a.e.:

(a) $\operatorname{rank}_{ss} \mathcal{G} = 1$, (b) $|W| = |\mathcal{B}^T| = 2$, (c) $\dim \mathcal{B} = 1$

(d) $\mathcal{B} \cong \mathbb{P}^1$ (e) There is an epi, $q\ \mathcal{G} \to PGL_2$ with

$(\ker q)^\circ = \mathcal{R}(\mathcal{G})$

Proof: Exercise, use Corollary 1.3, $\mathrm{Aut}_{grp}\, \mathbb{G}_m = \mathbb{Z}/2$ and $\mathrm{Aut}(\mathbb{P}^1) = PGL_2$ $\square$

Cor A: Let $G$ be reductive, $\mathrm{rank}_{ss}\, G = 1$, $T \subset G$ max. torus, $Z = Z(G)$. Then

(a) $G' = [G, G]$ is semisimple of dim 3

(b) $G = G'Z$, $G' \cap Z$ finite.

(c) $C_G(T) = T$, $Z(G) \subset T$

(d) If $\varphi : G \to PGL_2$ is epi, then $\ker \varphi = Z(G)$ $\square$

## 1.4. Weights and Roots

Let $T \cong \mathbb{G}_m^n$ be a torus

<u>Def</u>: (1) Let $V$ be a $T$-rep., $\chi \in X(T)$ and

$$V_\chi = \{ v \in V \mid tv = \chi(t) v \ \forall t \in T \}$$

Moreover $\Lambda(V) = \{ \chi \mid V_\chi \neq 0 \}$ is the set of <u>weights</u>

$$V = \bigoplus_\chi V_\chi$$

is called the weight space decomposition of $V$

(2) Let $T \subset G$ be a maxi .

$T$ acts on $\mathfrak{g}$ via Ad Then $\Phi(G,T) = \Lambda(\mathfrak{g}) - \{0\}$

is called the set of <u>roots</u> of $G$, relative to $T$

So we get

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g},T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = c_{\mathfrak{g}}(T) = \mathrm{Lie}(C_G(T))$

<u>Remark</u>: (1) It will also be useful to consider the following set of reduced roots: $\Phi_{red}(\mathfrak{g},T) \subset \Phi(\mathfrak{g},T)$, which is defined via

$$\mathfrak{g} = \mathrm{Lie}\Big(\bigcap_{\substack{T \subset B \\ B \subset \mathfrak{g} \\ \text{Borel}}} B\Big) \oplus \bigoplus_{\alpha \in \Phi_{red}(\mathfrak{g},T)}$$

Lecture 6

(2) The Weyl group $W = W(\mathcal{G}, T) = N_{\mathcal{G}}(T)/C_{\mathcal{G}}(T)$ acts on $X(T)$ and $Y(T)$, where $w \in N_{\mathcal{G}}(T)$ acts via

$$(w\chi)(t) = \chi(w^{-1} t w), \text{ for } \chi \in X(T), t \in T$$

$$(w\lambda)(z) = w\lambda(z)w, \text{ for } \lambda \in Y(T), z \in \mathbb{G}_m.$$

This action is compatible with $\langle , \rangle$, i.e. $\langle w-, - \rangle = \langle -, w- \rangle$.

Moreover, the action descends to $\Phi \subset X(T)$ $\square$

<u>Exercise</u>: Compute $\Phi \subset X(T)$ and the action of $W$ for $\mathcal{G} = GL_n$ $\square$

Cor: Let $G$ be reductive with $\operatorname{rank}_{ss} G = 1$.

$T$ a max. torus in $G$. Then

(1) $\ker(\mathrm{Ad}) = Z(G)$,

(2) $\Phi = \Phi_{red} = \{\pm\alpha\}$ for some $\alpha \in \Phi$.

(3) $\mathcal{B}^T = \{B_\alpha, B_{-\alpha}\}$ s.t. for $U_{\pm\alpha} = (B_{\pm\alpha})_u$,

$$\mathfrak{g}_{\pm\alpha} = \mathrm{Lie}\, U_{\pm\alpha}$$

(4) There is an isomorphism

$$u_\alpha : \mathfrak{g}_\alpha \longrightarrow U_\alpha$$

(5) $$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$ $\square$

Example: $G = GL_2 \supset T = \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}\right\}$.

$R(G) = Z(G) = \left\{\begin{pmatrix} x & \\ & x \end{pmatrix}\right\}$, $\quad GL_2/Z(G) = PGL_2$

$\varepsilon_i : T \to \mathbb{G}_m \qquad \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \mapsto z_i$

$\alpha = \varepsilon_1 - \varepsilon_2 : \qquad \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \mapsto z_1 z_2^{-1}$

$$\begin{array}{ccccc} \mathfrak{g} = & \mathfrak{g}_0 & \oplus \ \mathfrak{g}_\alpha & \oplus & \mathfrak{g}_{-\alpha} \\ & \| & \| & & \| \\ & \left\{\begin{pmatrix} a & \\ & d \end{pmatrix}\right\} & \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right\} & & \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right\} \\ & \| & \| & & \| \\ & \mathrm{Lie}\,T & \mathrm{Lie}\,U_\alpha & & \mathrm{Lie}\,U_{-\alpha} \end{array}$$

$B_\alpha = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\}$, $B_{-\alpha} = \left\{\begin{pmatrix} * & \\ * & * \end{pmatrix}\right\}$

$U_\alpha = \left\{\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right\}$, $U_{-\alpha} = \left\{\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right\}$

$u_\alpha : \mathbb{G}_a \to U_\alpha, \quad x \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$

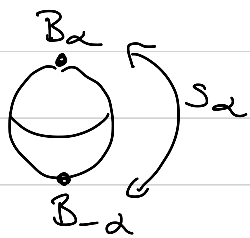
$$W = N_{\mathfrak{g}}(T)/T = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix}, \begin{pmatrix} & x \\ y & \end{pmatrix} \right\} / T$$

$$\wr\!\downarrow$$

$$S_2 = \{ e \ , \ s_\alpha \}$$

$$s_\alpha(\alpha) = -\alpha \ , \quad s_\alpha B_\alpha s_\alpha = B_{-\alpha}$$

$$\mathcal{B} \cong \mathbb{P}^1$$



## 1.5. Summoning $PGL_2$'s

<u>Definition</u>: Let $S \subset G$ be a torus in a conn. lin. alg. grp. Then $S$ is called regular, if $\mathcal{B}^S$ is finite. Otherwise $S$ is called singular $\square$

<u>Proposition</u>: Let $S \subset T \subset G$ be a torus and maximal torus in $G$. Then $S$ is regular iff $C = C_G(S)$ is solvable. In this case $\mathcal{B}^T = \mathcal{B}^S$ $\square$

<u>Proof</u>: Let $Y \subset X = \mathcal{B}^S$ be a connected component. Note that $C$ is connected (If $x \in C$, $s \in S$, both lie in a common Borel subgroup, which is connected).

Hence, $C$ acts on $Y$.

Claim: $C$ acts trasitively on $Y$.

Let $B \in Y$ and $Z = \{g \in G \mid gBg^{-1} \in Y\}$

We obtain a map

$$q: Z \times S \to B \to B/B_u = H, \quad (z,s) \mapsto zsz^{-1}.$$

where $H$ is also a torus. By the following Lemma $q$ is constant along $Z$. It follows that $zSz^{-1} \subset SB_u$ for all $z \in Z$. So, there is a $u \in B_u$, s.t. $nSn^{-1} = S$, for $n = uz \in N_G(S)$. So $CB \subset Z < NB$.

Since $N/C$ is finite, $\dim CB = \dim Z$. Both are connected, so $CB = Z$. Hence $C \cdot B = Z$ //Claim

We obtain $C/C\cap B \xrightarrow{\sim} Y \subset \mathcal{B}$ is complete

So $C\cap B \subset C$ is a Borel subgroup in $C$.

Hence :

$$C \text{ solvable} \Leftrightarrow C\cap B = C \Leftrightarrow \dim \mathcal{B}^S = 0 \Leftrightarrow \mathcal{B}^S \text{ finite}$$

The rest follows similarly $\square$

<u>Corollary</u>: $S$ is singular iff $S \subset T_\alpha$ for some

$\alpha \in \Phi_{red}$.

<u>Proof</u> Omitted. $\square$

**Lemma**: Let $\varphi: Z \times S \to H$ be a morphism where

(1) $S_{fin} = \{s \in S \mid |s| < \infty\} \subset S$ is dense

(2) $H_m = \{h \in H \mid h^m = 1\}$ is finite for each $m \in \mathbb{Z}_{>0}$

(3) $Z$ is connected

(4) For all $z \in Z$, $\varphi(z, -) : S \to H$ is a group hom.

Then $\varphi$ is constant along $Z$.

Pf: Let $s \in S$ with $s^m = 1$. Then $\varphi$ restricts to $Z \times \{s\} \to H_m$. Since $Z$ is connected, the image is a point. Hence $Z \times S_{fin} \to H$ is constant along $Z$. The statement follows from (1) $\square$

**Thm**: Let $\mathcal{G} \supset T$ be a conn. lin. alg. grp. with maximal torus. For $\alpha \in \Phi_{red}(\mathcal{G}, T)$, let

- $T_\alpha = \ker(\alpha : T \to \mathbb{G}_m)^\circ$
- $Z_\alpha = C_\mathcal{G}(T_\alpha)$
- $\mathcal{G}_\alpha = Z_\alpha / R(Z_\alpha)$

Then $\mathcal{G}_\alpha$ is a semisimple group of rank 1.

**Proof**: We have $T_\alpha \subset Z(Z_\alpha)^\circ \subset R(Z_\alpha)$.

Since $Z_\alpha$ is not solvable $R(Z_\alpha) \subsetneq Z_\alpha$ and

$\mathcal{G}_\alpha$ has rank $= \dim T/T_\alpha = 1$ $\square$

Lecture 7

## 1.6. Weyl chambers

Fix $G \supset T$ conn. lin. alg. grp. with max torus.

Let $\alpha \in \Phi_{red}$. Then, we obtain

$$\{e, s_\alpha\} = W(Z_\alpha, T_\alpha) = W_\alpha \subset W.$$

Moreover $Z_\alpha$ has two Borel subgroups $B_{-\alpha}, B_\alpha$,

s.t. $\mathfrak{g}_{\pm\alpha} = \operatorname{Lie} U_{\pm\alpha}$, $U_{\pm\alpha} = (B_{\pm\alpha})_u$.

We have $s_\alpha B_{-\alpha} s_\alpha = B_\alpha$.

Let $B \in \mathcal{B}^T$, then $B \cap Z_\alpha \in \{B_{-\alpha}, B_\alpha\}$, so

$\alpha$ induces a partition of $\mathcal{B}^T$.

Let $Y(T)_{reg} = \{\lambda : \mathbb{G}_m \to T \mid \lambda(\mathbb{G}_m) \text{ is regular}\}$

$$= \{\lambda \in \mathbb{G}_m \to T \mid \langle \alpha, \lambda \rangle \neq 0,\ \forall \alpha \in \Phi_{red}\}$$
$$(\Leftrightarrow \lambda(\mathbb{G}_m) \not\subset T_\alpha).$$

Let $\lambda \in Y(T)_{reg}$. Then, $\mathcal{B}^T = \mathcal{B}^{\lambda(\mathbb{G}_m)}$ and there is a unique maximal fixed point $B(\lambda) \in \mathcal{B}^{\lambda(\mathbb{G}_m)}$.

Here, maximal means that for all $B \in \mathcal{B}$, $\lambda(0)B \neq B(\lambda)$.

In fact $B(\lambda) \in \mathcal{B}^T$ is characterized by the property

$$B(\lambda) \cap Z_\alpha = B_\alpha \Leftrightarrow \langle \lambda, \alpha \rangle > 0.$$

For $B \in \mathcal{B}^T$, denote by

$$\mathcal{C}(B) = \{\lambda \in Y(T)_{reg} \mid B = B(\lambda)\}$$

Example: $\mathfrak{gL}_3 = \mathfrak{g} \supset T = \left\{\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}\right\}$

$X(T) = \langle \varepsilon_i : \mathrm{diag}(z_1, z_2, z_3) \mapsto z_i \rangle$

$\Phi = \Phi_{red} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}, \quad \alpha = \varepsilon_1 - \varepsilon_2, \beta = \varepsilon_2 - \varepsilon_3$

$T_\alpha = \left\{\begin{pmatrix} z_1 & & \\ & z_1 & \\ & & z_3 \end{pmatrix}\right\} \rightsquigarrow Z_\alpha = \left\{\begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}\right\},$

$R(Z_\alpha) = T_\alpha, \; \mathfrak{g}_\alpha = Z_\alpha / T_\alpha = \left\{\begin{pmatrix} \begin{bmatrix} * & * \\ * & * \end{bmatrix} & \\ & [*] \end{pmatrix}\right\} \cong \mathfrak{PgL}_2$

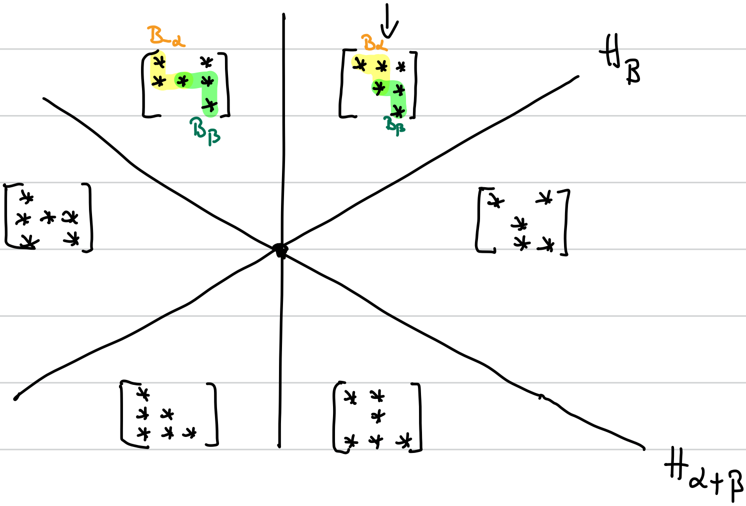
where $[\;]$ indicates elements in $\mathfrak{PgL}_2$.

$B_\alpha = \begin{pmatrix} * & * & \\ & * & \\ & & * \end{pmatrix} \subset Z_\alpha, \; B_{-\alpha} = \begin{pmatrix} * & & \\ * & * & \\ & & * \end{pmatrix}$

$\rightsquigarrow$ See $\mathfrak{gL}_2$ example 1.4

$H_\alpha = \{\lambda \in Y(T) \mid \langle \alpha, \lambda \rangle = 0\}$

$H_\alpha$ $\quad B(\lambda)$, $\langle \lambda, \alpha \rangle > 0$, $\langle \lambda, \beta \rangle > 0$




$\bullet = H_\alpha \cap H_\beta = \langle \varepsilon_1^* + \varepsilon_2^* + \varepsilon_3^* \rangle \subset Y(T)$

$= Y(Z(g))$

## 1.7. The unipotent radical

Recall that $I(T) = \left(\bigcap_{B \in \mathcal{B}^T} B\right)^\circ$

<u>Thm</u>: Let $G \supset T$ be a conn. lin. alg. grp. with max. torus.

Then $R_u(G) = I(T)_u$

<u>Pf</u>: Let $U = I(T)_u$. Clearly $R_u(G) \subset U$.

So we need to show $U$ is normal $G$.

Now $G = \langle B \in \mathcal{B}^T \rangle$, so it suffices to show $B \subset N_G(U)$.

Choose $B \in \mathcal{B}^T$. Let $A = \langle C_B(S) \mid S \subset T \text{ torus, } \dim T/S = 1 \rangle$

We claim that $A = B$. Suffices to show $\mathfrak{a} = \mathfrak{b}$. For this,

we compare $\mathfrak{a}_\lambda$ and $\mathfrak{b}_\lambda$ for $\lambda \in X(T)$.

If $\lambda = 0$, we have

$$\mathfrak{b}_0 = C_{\mathfrak{b}}(T) = \mathrm{Lie}(C_B(T)) \subset \mathrm{Lie}(C_B(S)) \subset \mathfrak{a}$$

For $\lambda \neq 0$, let $S = (\ker \lambda)^\circ$, then, similarly,

$$\mathfrak{b}_\lambda = C_{\mathfrak{b}}(S) \subset \mathfrak{a}.$$

So, it suffices to show $C_B(S) \subset N_G(\mathfrak{u})$, $S \subset T$ torus, $\dim(T/S) = 1$. If $S$ is regular, $C_B(S) = C_G(S)$ (1.5. Proposition) and $C_G(S)$ is contained in all Borel subgroups in $\mathcal{B}^S = \mathcal{B}^T$. (1.1 Facts)

So $C_B(S) \subset N_G(\mathfrak{u})$.

Now, let $S$ be singular, so $S = (\ker \alpha)^\circ = T_\alpha$ for $\alpha \in \Phi_{red}$

(1.5 Corollary). Then

$$C_B(T_\alpha) = B \cap C_G(T_\alpha) = B \cap Z_\alpha \in \{B_\alpha, B_{-\alpha}\}.$$

W.l.o.g. $C_B(T_\alpha) = B_\alpha$, so $B = B(\lambda)$ with

$\langle \alpha, \lambda \rangle > 0$ for some $\lambda \in Y_{reg}(T)$.

In total, we can sort the weights in $\mathfrak{g}$

$$\underbrace{-\alpha, -\beta, \dots}_{\mathfrak{g}/\mathfrak{b}}, \quad \underset{\mathfrak{t}}{0}, \quad \underbrace{\alpha, \beta, \dots}_{\mathfrak{b}_u/\mathfrak{u}}, \quad \underset{\mathfrak{u}}{\gamma, \delta, \dots}$$

$\in \Phi_{red}$

Let $H = \bigcap_{\langle \alpha, \lambda \rangle > 0} B(\lambda)_u \supset U$. Then $B_\alpha \subset H$.

Let $\beta$ be a weight of $\mathfrak{h}/\mathfrak{u}$. Assume $\alpha \neq \beta$.

Then $\beta \neq 0$ and $\beta \neq -\alpha$ (Since $(B_{-\alpha})_u \not\subset B(\lambda)$ if

$\langle \alpha, \lambda \rangle > 0$). Since $\alpha, \beta$ are are non-proportional,

There is a $\lambda \in Y(T)_{reg}$, s.t., $\langle \alpha, \lambda \rangle > 0$, $\langle \beta, \lambda \rangle < 0$

Then $B(\lambda) \cap Z_\beta = B_{-\beta}$ with $B(\lambda) \supset H$.

Sine $-\beta$ does not occur in $\mathfrak{h}$, we have

$Z_\beta \cap H = C_H(T_\beta) = 1$, so $C_{\mathfrak{h}}(T_\beta) = \mathfrak{h}_\beta = 0$ ↯

Hence $\mathfrak{h}/\mathfrak{u} = \mathfrak{g}_\alpha$ and $\dim H/U = 1$. Since

$H$ is unipotent it is nilpotent. So $U \trianglelefteq H$. □

<u>Corollary</u>: Let $G$ be reductive, $T \subset G$ max. torus.

(a) For a torus $S \subset T \subset G$, $C_G(S)$ is reductive

and $C_G(S) = S$ if $S$ is regular

(b) $\Phi = \Phi_{red}$ , $\Phi = -\Phi$

(c) $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ , $\dim \mathfrak{g}_\alpha = 1$

(d) $Z_\alpha$ is reductive and $\mathfrak{z}_\alpha = \mathfrak{t} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$,

the $Z_\alpha$ generate $G$.

(e) $Z(G) = \bigcap_{\alpha \in \Phi} T_\alpha$, $X(Z(G)) = X(T)/\mathbb{Z}\Phi$.

(f) For each $B \in \mathcal{B}$ there is a $B^- \in \mathcal{B}$ with

$B^- \cap B = T$, $\mathfrak{g} = \mathfrak{b}^- + = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$.

<u>Proof</u>: Sketch + Exercise

(a) Let $C = C_{\mathcal{G}}(S)$. Then $R_u(C) = I(T)_u \cap C = R_u(\mathcal{G}) = \{e\}$,

so $C$ is reductive.

If $S$ is regular, $C$ is solvable (1.5 Proposition).

But reductive + solvable = torus, so $C = T$.

(b), (d) immediate.

(e) $\bigcap T_\alpha \subseteq \bigcap C_{\mathcal{G}}(Z_\alpha) \subseteq Z(\mathcal{G}) \subset \bigcap Z(Z_\alpha) = \bigcap T_\alpha$

$Z_\alpha$ generate $\mathcal{G}$

(f) Let $B = B(\lambda)$, set $B^- = B(-\lambda)$

Lecture 8

## 1.8 Root subgroups, $SL_2$ and the root datum.

<u>Thm</u>: Let $\mathcal{G}$ be reductive with max. torus $T$,

$\alpha \in \underline{\Phi} = \Phi(\mathcal{G}, T)$.

(a) There is a unique $T$-stable connected subgroup

$$U_\alpha < B_\alpha \subset Z_\alpha \subset \mathcal{G} \quad \text{with } \operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha.$$

(b) If $w = [n] \in W$, $n U_\alpha n^{-1} = U_{n(\alpha)}$.

(c) There is an isomorphism

$$u_\alpha : \mathcal{G}_a \to U_\alpha, \text{ s.t.}$$

$$t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t) x)$$

(d) There is a $(2:1$ or $1:1)$ map

$$SL_2 \to [Z_\alpha, Z_\alpha] \to Z_\alpha \to G$$

such that

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mapsto u_\alpha(x) \qquad \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix} \mapsto u_{-\alpha}(x)$$

This yields an isom. of Lie algebras.

$$\mathfrak{sl}_2 \to \mathfrak{sl}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

<u>Def + Remark</u>: For $\alpha \in \Phi$, we obtain a unique cocharacter $\alpha^\vee : \mathbb{G}_m \to [Z_\alpha, Z_\alpha] \cap T \to T$, such that $\langle \alpha, \alpha^\vee \rangle = 2$. We call $\alpha^\vee$ the coroot associated to $\alpha$ and denote by

$$\underline{\Phi^\vee} = \{\alpha^\vee \mid \alpha \in \underline{\Phi}\} \subset Y(T)$$

the set of coroots.

To $G \supset T$ we hence associate the following data:

$$(X(T), \Phi, Y(T), \Phi^\vee, \langle\, , \rangle, {}^\vee : \Phi \xrightarrow{\sim} \Phi^\vee)$$

which we call the <u>root datum</u> associated to $G \supset T$.

Often $\langle\, , \rangle$ and the bijection $^\vee$ are dropped from the notation.

## 1.9. An extended example: $Sp_4$

Let $G = Sp_4 = \{ A \in GL_4 \mid A J_4 A^{tr} = J_4 \}$

where $J_4 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$.

Then we get

$$\mathfrak{g} = \mathfrak{sl}_4 = \{ X \in \mathfrak{gl}_4 \mid X J_4 + J_4 X^{tr} = 0 \}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & -A^{tr} \end{pmatrix} \;\middle|\; B^{tr} = B,\ C^{tr} = C \right\}$$

and $\dim Sp_4 = 2(2)^2 + 2 = 10$

And the maximal torus

$$T = \left\{ \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_1^{-1} & \\ & & & x_2^{-1} \end{pmatrix} \right\} \subset G$$

$X(T) = \langle \varepsilon_1, \varepsilon_2 \rangle$, $Y(T) = \langle \varepsilon_1^*, \varepsilon_2^* \rangle$.

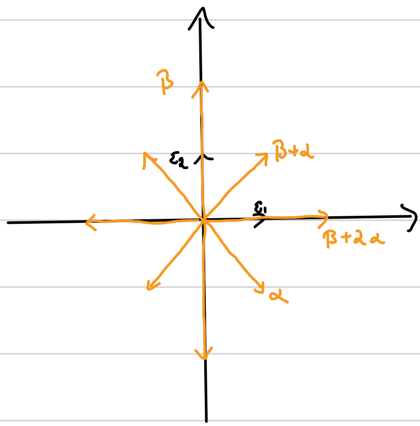
| $\gamma$ | $U_\gamma$ | $\gamma^\vee(\mathbb{G}_m)$ | $\gamma^\vee$ |
|---|---|---|---|
| $\alpha = \varepsilon_1 - \varepsilon_2$ | $\left(\begin{array}{cc\|cc} 1 & a & & \\ & 1 & & \\ \hline & & 1 & \\ & & -a & 1 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} x & & & \\ & x^{-1} & & \\ \hline & & x^{-1} & \\ & & & x \end{array}\right)$ | $\alpha^\vee = \varepsilon_1^* - \varepsilon_2^*$ |
| $\beta = 2\varepsilon_2$ | $\left(\begin{array}{cc\|cc} 1 & & & \\ & 1 & & a \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} 1 & & & \\ & x & & \\ \hline & & 1 & \\ & & & x^{-1} \end{array}\right)$ | $\beta^\vee = \varepsilon_2^*$ |
| $\alpha+\beta = \varepsilon_1 + \varepsilon_2$ | $\left(\begin{array}{cc\|cc} 1 & & & a \\ & 1 & a & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} x & & & \\ & x & & \\ \hline & & x^{-1} & \\ & & & x^{-1} \end{array}\right)$ | $(\alpha+\beta)^\vee = \varepsilon_1^* + \varepsilon_2^*$ |
| $2\alpha+\beta = 2\varepsilon_1$ | $\left(\begin{array}{cc\|cc} 1 & & a & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ | $\left(\begin{array}{cc\|cc} x & & & \\ & 1 & & \\ \hline & & x^{-1} & \\ & & & 1 \end{array}\right)$ | $(2\alpha+\beta)^\vee = \varepsilon_1^*$ |

and $U_{-\gamma} = U_\gamma^{tr}$. For example:

$$Z_\alpha = \langle T, U_\alpha, U_{-\alpha} \rangle = \left(\begin{array}{c\|c} A & \\ \hline & A^{-tr} \end{array}\right) \qquad [Z_\alpha, Z_\alpha] = SL_2$$

$$Z_{\alpha+\beta} = \langle T, U_{\pm(\alpha+\beta)} \rangle = \left(\begin{array}{cc\|cc} a & & & b \\ & a & b & \\ \hline & c & d & \\ c & & & d \end{array}\right) \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in SL_2$$

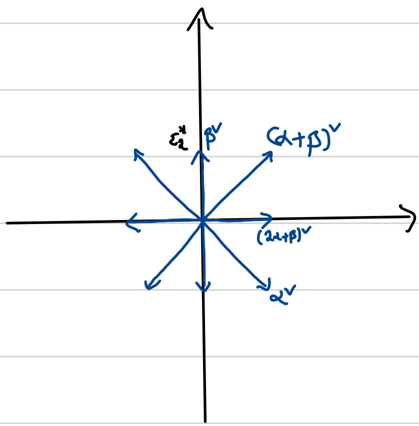
$\Phi \subset X(T)$



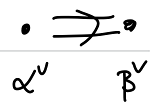

Type $C_2$


$$W = \underset{\langle s_\beta\rangle \times \langle s_{\beta+2\alpha}\rangle}{\underset{\shortparallel}{(C_2)^2}} \rtimes \underset{\langle s_\alpha\rangle}{\underset{\shortparallel}{S_2}}$$

$\Phi^\vee \subset Y(T)$




Type $B_2$




Observations: (a) $X(Z(g)) = X(T)/\mathbb{Z}\Phi = \mathbb{Z}/2$

So $Z(g) = \mu_2 \rightsquigarrow$ $\left|\begin{array}{cc|cc} -1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right|$

(b) There is a <u>root string</u> of the form

$$\beta, \ \beta+\alpha, \ \beta+2\alpha$$

Observe $[\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}] = \mathfrak{g}_{\alpha_1+\alpha_2}$

Let $W = \bigoplus_n \mathfrak{g}_{\beta+n\alpha}$ $\rightsquigarrow$ $W$ is a $Z_\alpha$ rep. $\rightsquigarrow$ $SL_2$ rep.

$$\mathrm{Ad}\left(\begin{array}{c|c} A & \\ \hline & A^{-t} \end{array}\right)\left(\begin{array}{c|c} & B \\ \hline & \end{array}\right) = \left(\begin{array}{c|c} & ABA^{t} \\ \hline & \end{array}\right)$$

So $W$ is the three dim. irred. $SL_2$-rep. $S^2(k^2)$

Then length of the root string $=$ dim $W = |\langle \underbrace{\beta}_{\text{lowest weight}}, \alpha^\vee\rangle| + 1$

(c) $[Z_\beta, Z_{\beta+2\alpha}] = 1$, since $\beta+2\alpha \pm \beta \notin \Phi$

<u>Exercise</u>: Do $SO_4$ in the same detail. Also: $Sp_{2n}$!

Lecture 9

## 1.10 Interlude: Rep. th. of $SL_2$

Denote by $V = k^2$, the std. rep. of $SL_2$.

Write $\quad sl_2 = \left\langle E = \begin{pmatrix} & 1 \\ & \end{pmatrix}, H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, F = \begin{pmatrix} & \\ 1 & \end{pmatrix} \right\rangle$

So $[E,F] = H$, $[H,E] = 2E$, $[H,F] = -2F$.

For $n \geq 0$, write $L(\lambda) = S^\lambda(V) = \{ f \in k[x,y] \mid f \text{ hom. of degree } \lambda \}$

$$L(\lambda): \begin{tikzcd} \bullet \arrow[loop above, "-\lambda"] \arrow[r, shift left, "1"] & \bullet \arrow[loop above, "-\lambda+2"] \arrow[l, shift left, "\lambda"] \arrow[r, shift left, "2"] & \bullet \arrow[loop above, "-\lambda+4"] \arrow[l, shift left, "\lambda-1"] & \cdots & \bullet \arrow[loop above, "\lambda-4"] \arrow[r, shift left, "\lambda-1"] & \bullet \arrow[loop above, "\lambda-2"] \arrow[l, shift left, "2"] \arrow[r, shift left, "\lambda"] & \bullet \arrow[loop above, "\lambda"] \arrow[l, shift left, "1"] \end{tikzcd}$$

$$\xleftarrow{F} \quad \circlearrowright^{H} \quad \xrightarrow{E}$$

$$\{\text{f.d. } SL_2\text{-rep.}\} \overset{1:1}{\longleftrightarrow} \{\text{f.d. } sl_2\text{-rep.}\} \rightsquigarrow \text{semisimple}$$

$$W = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0} \\ \| \\ X(T)/W = X(T)_+}} L(\lambda)^{n_\lambda}$$

## 1.11 Abstract Root Data

<u>Defn</u>: An abstract root datum is a tuple

$$(X, \Phi, Y, \Phi^\vee, (\ )^\vee, \langle\ ,\ \rangle)$$

such that:

(1) $X$ and $Y$ are f.g. free abelian groups,

$\langle\ ,\ \rangle : X \otimes Y \to \mathbb{Z}$ is a perfect pairing

(2) $\Phi \subset X$, $\Phi^\vee \subset Y$ are finite subsets and

$(\ )^\vee : \Phi \to \Phi^\vee$ a bijection

(3) $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $r_\alpha : X \to X$, $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$

is an automorphism of the root datum

(5) $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

<u>Def B</u>: An abstract root system is a tuple

$$(V, \Phi, \Phi^\vee, (\ )^\vee)$$

where

(1) $V$ is a real vector space

(2) $\Phi \subset V$, $\Phi^\vee \subset V^*$ are finite subsets and

$(\ )^\vee : \Phi \to \Phi^\vee$ a bijection

(3) For $\alpha, \beta \in \Phi$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $\tau_\alpha : V \to V$, $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$ and
$\tau_{\alpha^\vee} : V^* \to V^*$, $\lambda \mapsto \lambda - \langle \alpha, \lambda \rangle \alpha^\vee$
permute $\Phi$ and $\Phi^\vee$.

(5) If $\alpha$ and $c\alpha \in R$, then $c = \pm 1$

Remark: (1) There is a duality for root data:

$$(X, \Phi, Y, \Phi^\vee) \longleftrightarrow (Y, \Phi, X, \Phi^\vee)$$

(2) $$(X, \Phi, Y, \Phi^\vee) \longmapsto (V, \Phi, \Phi^\vee, (\ )^\vee)$$
$$V = \langle \Phi \rangle_{\mathbb{Z}} \otimes \mathbb{R}$$

yields a root system. In particular, there is a notion of irreducible root system, bases, positive roots, Dynkin diagram, Weyl group, ..

(3) Root systems are equivalently defined as $(E, \Phi)$ where $E = (E, (\ ,\ ))$ is a Euclidean v.s.

Then $\alpha^\vee = 2\dfrac{(-,\alpha)}{(\alpha,\alpha)}$.

## 1.12. Root datum of a reductive group

<u>Thm</u>: Let $G \supset T$ be a reductive group with max. torus

Then $(X(T), \Phi, Y(T), \Phi^\vee)$ is an (abstract)

root datum. Moreover the Weyl group $W(G,T)$

and $W(R, \Phi)$ coincide

<u>Proof</u>: We have to show 5 Axioms

(1) – (3) Clear by definition.

(4) Let $\alpha \in \Phi$. Then we obtain $s_\alpha \in W(Z_\alpha, T) \subset W(G,T)$

and $\tau_\alpha : \chi \mapsto \chi - \langle \chi, \alpha^\vee \rangle \alpha^\vee$.

We know that $s_\alpha$ induces an isomorphism of the root datum.

Let $V = SL_2\, \mathfrak{g}_\beta = \mathfrak{sl}_2\, \mathfrak{g}_\beta \subset \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$.

Then this is an irreducible representation.

Consider
$$\begin{aligned} X(T) &\longrightarrow X(\alpha^\vee(\mathbb{G}_m)) = \mathbb{Z} \\ \chi &\longmapsto \chi|_{\alpha^\vee(\mathbb{G}_m)} = \langle \chi, \alpha^\vee \rangle. \end{aligned}$$

So $V_{\chi|\alpha(\mathbb{G}_m)} = V_{\langle \chi, \alpha^\vee \rangle}$ and

$$s_\alpha(V_{\langle \chi, \alpha^\vee \rangle}) = V_{-\langle \chi, \alpha^\vee \rangle}.$$

Hence $\langle s_\alpha(\chi), \alpha^\vee \rangle = -\langle \chi, \alpha \rangle$.

Moreover, if $\langle \chi, \alpha^\vee \rangle = 0$, we have $s_\alpha(\chi) = \chi = \sigma_\alpha(\chi)$

Hence $s_\alpha = \sigma_\alpha$.

Moreover, we have

$$\begin{tikzcd} W(\mathfrak{g}, T) \arrow[r, "\sim"] \arrow[d, "\text{torsor}"] & W(R, \Phi) \arrow[d, "\text{torsor"] \\ \mathcal{B}^T \arrow[r, leftrightarrow, "1:1"] \arrow[u, Rightarrow] & \text{Weyl chambers in } Y(T)_{reg} \end{tikzcd}$$

(5) Exercise (Use $\mathfrak{sl}_2$-rep. th.)

Remark:

We obtain the following chain of $W$-torsors

$$
\begin{array}{ccc}
\mathcal{B}^T & & B(\lambda) \\
\uparrow & & \begin{array}{c}\uparrow\\[-4pt]\vert\end{array} \\
\text{Weyl chambers in } Y(T)_{reg} & & C(\lambda) \\
\downarrow & & \begin{array}{c}\vert\\[-4pt]\downarrow\end{array} \\
\text{Connected components of } Y(T)_{\mathbb{R}} - \bigcup_{\alpha\in\Phi} H_\alpha & & \text{"}C(\lambda)_{\mathbb{R}}\text{"} \\
\downarrow & & \begin{array}{c}\vert\\[-4pt]\downarrow\end{array} \\
\text{Choice of positive roots} & & \Phi^+ = \{\alpha\in\Phi \mid \langle\alpha,\lambda\rangle > 0\} \\
\downarrow & & \begin{array}{c}\vert\\[-4pt]\downarrow\end{array} \\
\text{Choice of simple roots} & & \Delta = \{\alpha\in\Phi^+ \mid \alpha\notin \mathbb{Z}_{\geq 0}\Phi^+\setminus\{\alpha\}\} \\
\downarrow & & \begin{array}{c}\vert\\[-4pt]\downarrow\end{array} \\
\text{Choice of simple reflections} & & S = \{s_\alpha \mid \alpha\in\Delta\} \subset W
\end{array}
$$

In particular $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$. $\square$

## 1.13 Weyl group and Type

Recall that the Dynkin diagram $\Gamma$ arises from $\Delta$ by

Vertices $= \Delta$

$\#$ edges from $\alpha$ to $\beta = d_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle \langle \beta^\vee, \alpha \rangle \in \{0, 1, 2, 3\}$ for $\alpha \neq \beta$

$0 \to A_1 \times A_1$, $1 \to A_1$, $2 \to B_2, C_2$, $3 \to G_2$

For $\langle \alpha, \beta^\vee \rangle \langle \beta^\vee, \alpha \rangle > 1$ and if $\langle \alpha, \beta^\vee \rangle = -1$ put an arrow

$$\underset{\alpha}{\bullet} \Leftarrow \underset{\beta}{\bullet}$$

The Weyl group is a Coxeter group with presentation

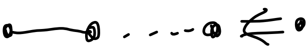
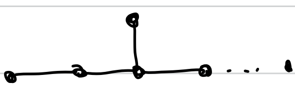
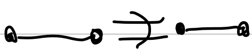
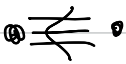
$$W = \langle s_\alpha \in S \mid s_\alpha^2 = 1, \ \underbrace{s_\alpha s_\beta \cdots}_{m_{\alpha,\beta}} = \underbrace{s_\beta s_\alpha \cdots}_{m_{\alpha\beta}}$$

where

| $m_{\alpha\beta}$ | $d_{\alpha,\beta}$ |
|---|---|
| 2 | 0 |
| 3 | 1 |
| 4 | 2 |
| 6 | 3 |

# Lecture 10

For an irreducible root datum, the <u>type</u> is the type of the associated root system.

| | | | |
| --- | --- | --- | --- |
| Classical |  | $A_n$ | $GL_{n+1}$, $SL_{n+1}$, $PGL_{n+1}$ (dual) |
| |  | $B_n$ | $SO_{2n+1}$ (dual) |
| |  | $C_n$ | $Sp_{2n}$ |
| |  | $D_n$ | $SO_{2n}$ |
| Exceptional |  | $E_6, E_7, E_8$ | |
| |  | $F_4$ | |
| |  | $G_2$ | |

# Sketch:

$$\begin{tikzcd}
\{\text{root data}\}/\sim \arrow[r] \arrow[d, leftrightarrow, "1:1"', "\text{To show}"] & \{\text{root systems}\}/\sim \arrow[d, leftrightarrow] \\
\{\text{reductive groups over } k\}/\sim \arrow[r] & \{\text{semisimple Lie alg.}\}/\sim \\
G \arrow[r, mapsto] & \mathrm{Lie}([G,G]) = \mathrm{Lie}(G/Z(G))
\end{tikzcd}$$

$\uparrow$

forgets center + $\underset{\text{finite covers}}{\text{isogeny}}$

$GL_2$ vs. $SL_2$ $\qquad$ $PGL_2$ vs. $SL_2$

## 1.14 Generators, Automorphisms and simple constituents.

<u>Lemma</u>: Let $G \supset T$ be a reductive group with max. torus. Let $\Delta \subset \Phi$ be a set of simple roots. Then $G$ is generated by $U_{\pm\alpha}, \alpha \in \Delta$, and $T$.

<u>Proof</u>: For $\alpha \in \Delta$, $Z_\alpha = C_G((\ker\alpha)^\circ)$ is generated by $U_{\pm\alpha}$ and $T$ (Theorem 1.3, 1.8).

Let $H = \langle Z_\alpha \mid \alpha \in \Delta \rangle$. Then $H$ contains $N_G(T)$, since $W$ is generates by $s_\alpha \in N_{Z_\alpha}(T)/T$. Hence $\mathfrak{g}_{\sigma(\alpha)} \subset \mathfrak{h}$ for all $\alpha \in \Delta$, $\sigma \in W$. All elements in $\Phi$ arise this way. So $\mathfrak{h} = \mathfrak{g}$ which forces $H = G$ $\square$

**Prop:** Let $G \supset B \supset T$ be a reductive group with Borel and maximal torus. Let $\Delta \subset \Phi$ be the sets of simple roots. There is a natural morphism $\varphi$,

$$\begin{array}{l} D = \{\sigma \in \mathrm{Aut}(G) \mid \sigma(T) = T, \sigma(B) = B\}, \\ \big\downarrow \\ \Gamma = \{\sigma \in \mathrm{Aut}(X(T), \Phi) \mid \sigma(\Delta) = \Delta\} \end{array}$$

$$(\varphi(\sigma))(\alpha) = \beta \iff \sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta.$$ Then:

(a) $\mathrm{Aut}(G) = \mathrm{Int}(G) D$ and

(b) $\varphi : \mathrm{Aut}/\mathrm{Int}(G) \hookrightarrow \Gamma$ is injective.

Proof: (a) Let $\sigma \in Aut(\mathcal{G})$. Then there is $x \in \mathcal{G}$, s.t.

$x\,\sigma(B)x^{-1} = B$, and $y \in B$, s.t. $yx\sigma(T)x^{-1}y^{-1} = T$.

Hence $Int(xy)\,\sigma \in D$.

(b) Let $\sigma \in D$, s.t. $\varphi(\sigma) = e \in \Gamma$.

Since $\varphi(\sigma) = e : X(T) \to X(T)$, $\sigma|_T = e$. Moreover

$\sigma(U_\alpha) = U_\alpha$ for all $\alpha \in \Delta$. Hence, we obtain

$$Aut(U_\alpha) \longrightarrow \mathbb{G}_m,\ \varphi|_{U_\alpha} \longmapsto c_\alpha.$$

Since $\Delta$ is linearly independent, we find $t \in T$, s.t.

$\alpha(t) = c_\alpha$. Let $\sigma' = Int(t)\,\sigma$. Then $\sigma'|_{U_\alpha} = e$ $\forall \alpha \in \Delta$

and $\sigma'|_T = e$. By the previous lemma, $\sigma' = e$. $\square$

Thm: Let $G$ be semisimple and $\mathcal{S}$ the collection of minimal connected normal subgroups of positive dimension. Then

(a) $[H, H'] = 1$ for all $H \neq H' \in \mathcal{S}$.

(b) $|\mathcal{S}| = n < \infty$ and $\pi : H_1 \times \cdots \times H_n \to G$ is surjective with finite kernel for all enumerations $\mathcal{S} = \{H_1, \ldots, H_n\}$

(c) Let $G' \subset G$ be any connected subgroup of pos. dim. Then $G'$ is the product of all groups in $\mathcal{S}_{G'} = \{H \in \mathcal{S} \mid H \subset G\}$, and $H \in C_G(G')$ $\forall H \in \mathcal{S} \setminus \mathcal{S}_{G'}$.

(d) $G = (G, G)$

(e) $G$ is generated by $U_{\pm\alpha}$, $\alpha \in \Delta$.

Proof: (1) $[H, H']$ is connected normal, so trivial by minimality.

(2) Let $\{H_1, \dots, H_m\} = \mathcal{S}' \subset \mathcal{S}$ be a finite subset.

By (a) $\pi : H_1 \times \cdots \times H_m \to \mathcal{G}$ is a hom. of groups.

and $H = \operatorname{im} \pi \trianglelefteq \mathcal{G}$. We have $R(\mathcal{G}') \subseteq R(\mathcal{G}) = \{e\}$, so

$H$ is semisimple. By (a) $H' \subset C_{\mathcal{G}}(\mathcal{G}')$ for all $H' \in \mathcal{S} \setminus \mathcal{S}'$.

and hence $H' \cap H$ is finite by minimality of $H'$.

Similarly $\ker \pi$ is finite by minimality. Hence $\mathcal{S}$ is finite by dimension. Let $\mathcal{S}' = \mathcal{S} = \{H_1, \dots, H_n\}$.

We show that $H = \mathcal{G}$.

We obtain

$$\begin{array}{ccccccc} 1 \to & C_g(H) & \to & g & \longrightarrow & \mathrm{Aut}(H) \\ & \downarrow & & \downarrow & & \downarrow \\ 1 \to & C_g(H)H/H & \to & g/H & \longrightarrow & \mathrm{Aut}(H)/\mathrm{Inn}(H) \end{array}$$

$\uparrow$ finite (Exercise)

Let $H' = (C_g(H))^\circ$. So $H'H \subset g$ has finite index.

By connectedness $H'H = g$. Hence $H'$ is normal in $g$

which forces $H' = \{e\}$.

(c) similar to (a).

(d) $[H, H] \trianglelefteq g$ for all $H \in \mathcal{S}$, so $[H, H] = H$ since

$H$ minimal and $g$ semisimple. So

$$[g, g] = \prod_{H \in \mathcal{S}} [H, H] = \prod_{H \in \mathcal{S}} H = g.$$

(e) Exercise

$\square$

<u>Corollary</u>: (a) The collection $\mathcal{S}$ from Theorem 1.14 is in bijection with the irreducible components of the root system.

(b) $\mathcal{B}_G = \prod_{H \in \mathcal{S}} \mathcal{B}_H$.

(b) For $G$ reductive, $G = Z(G)[G,G]$.

<u>Proof</u>: Exercise. $\square$

<u>Def</u>: A semisimple group $G$ with irred. root system is called <u>almost simple</u>. $\square$

Hence, if we want to understand the geometry of flag varieties, it suffices to consider almost simple groups

Lecture 11

## 1.15. The exchange condition

Bourbaki, Lie groups and Lie algebras, Chapters 4–6

<u>Def:</u> Let $W$ be a group generated by a set $S = S^{-1}$, $1 \notin S$

(1) The length $\ell(w) = \ell_S(w)$ is the smallest integer, such that $w$ is a product of $\ell(w)$ elements in $S$.

(2) A sequence $\underline{w} = (s_1, \dots, s_n) \in S^n$ is called a <u>reduced expression</u> of $w$, if $s_1 \cdots s_n = w$ and $n = \ell(w)$ o

(3) The Bruhat order $\leq$ on $W$ is defined by

$x \leq y \iff$ a reduced expression of $x$ is a subsequence of a reduced expression of $y$. $\square$

Thm (Exchange condition) $W$ has a presentation of the form $W = \langle s \in S \mid s^2 = 1,\ st^{m(s,t)} = 1 \rangle$ for suitable $m(s,t) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, i.e. is a Coxeter group, if and only if for each $w \in W$ with reduced expression $(s_1, \dots, s_n)$ and $s \in S$ with $\ell(sw) \leq \ell(w)$ there is a $1 \leq j \leq n$, s.t.,

$$s s_1 \cdots s_{j-1} = s_1 \cdots s_j s_{j+1}$$

Exchange!

Proof: Omitted. $\square$

**Corollary**: Let $W \supset S$ be a Coxeter group, $w \in W$ with reduced expression $(s_1, \dots, s_n)$ and $s \in S$. Then, there are two cases:

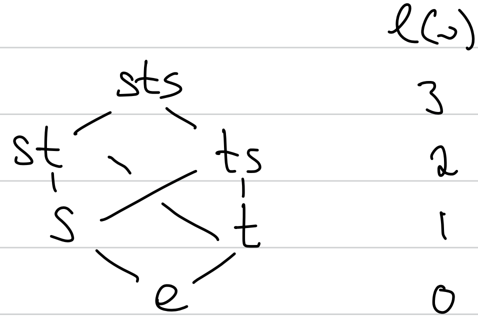
(1) $\ell(sw) = \ell(w) + 1$, and $(s, s_1, \dots, s_n)$ is a reduced expression of $sw$, or

(2) $\ell(sw) = \ell(w) - 1$, and there is a $1 \leq j \leq n$, s.t. $(s_1, \dots, \hat{s}_j, \dots, s_n)$ and $(s, s_1, \dots, \hat{s}_j, \dots, s_n)$ are reduced expressions of $ws$ and $w$, respectively.

*Proof*: Exercise. Hint: First show that $|\ell(w)-\ell(w')| \leq \ell(ww'^{-1})$ for all $w, w' \in W$. Use this for $w' = sw$. Then use exchange prop. □

*Example*: Let $W = \langle \underset{(1,2)}{\underset{\shortparallel}{s}}, \underset{(2,3)}{\underset{\shortparallel}{t}} \mid s^2 = t^2 = 1, \ sts = tst \rangle \cong S_3$

Then the Bruhat order and length are given by




□

*Exercise*: For $W = S_n = \langle \underset{(1,2)}{\underset{\shortparallel}{s_1}}, \dots, \underset{(n-1,n)}{\underset{\shortparallel}{s_{n-1}}} \rangle$, determine $m_{st}$.

Show that $\ell(w) = |\{ i<j \mid w(i) > w(j) \}|$. □

Remark: By a Theorem of Matsumoto in a Coxeter group any two reduced expressions of the same element are linked via braid relations, $sts\cdots \to tst\cdots$

## 1.15 Tits system, Bruhat decomposition and Coxeter group

In this section, we deal with abstract groups.

**Definition**: A Tits system is a tuple

$$(\mathcal{G}, B, N, S)$$

where $\mathcal{G}$ is a group with subgroups $B, N$ and

$S \subset W = N/_{B \cap N}$ is a subset, such that:

(1) $B \cup N$ generates $\mathcal{G}$, $B \cap N \trianglelefteq N$

(2) $S$ generates $W = N/(B \cap N)$ and $|s| = 2 \ \forall s \in S$

(3) $sBw \subset BwB \cup BswB$ for $s \in S$, $w \in W$.

(4) $sBs \not\subset B$ for all $s \in S$. $\square$

Think: $(\mathcal{G}, B, N, S)$
$\uparrow$ reductive, $\uparrow$ Borel, $\uparrow$ $N_{\mathcal{G}}(T)$, $\uparrow$ simple reflections in $W$ associated to $B$. $\square$

Fix a Tits system $(\mathcal{G}, B, N, S)$.

<u>Proposition</u>: Let $s \in S$, $w, w' \in W$ and $\ell = \ell_S : W \to \mathbb{Z}_{\geq 0}$ the length function. Then

(1) $Bww'B \subset BwBw'B$, $Bw^{-1}B = (BwB)^{-1}$.

(2) $BsBwB \subset BswB \cup BwB$

(3)
$$BsBwB = \begin{cases} BswB & \text{if } \ell(sw) = \ell(w)+1 \\ BwB \cup BswB & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

(4) $BwBsB = BwBwBwsB, \ldots$

(5) For $s_1, \ldots, s_n \in S$, we get

$$B s_1 \cdots s_n B w B \subset \bigcup_{1 \le i_1 < \cdots < i_m \le p} B s_{i_1} \cdots s_{i_m} w B$$

Proof: Exercise $\square$

<u>Thm A</u> (a) For $I \subset S$, let $W_I = \langle s \in I \rangle$, $P_I = BW_IB$.

Then $P_I \subset G$ is a subgroup, called <u>parabolic subgroup</u>

(b) $G = BWB = \bigoplus_{w \in W} BwB$. This is called the

<u>Bruhat decomposition</u> of $G$.

<u>Pf</u>: (a) Use previous Proposition (5)

(b) By (a) $BWB \subset G$ is a subgroup. Since $N, B \subset BWB$

generate $G$, $BWB = G$. It remains to show

$w \neq w' \Rightarrow BwB \neq Bw'B$. Let $\ell = \ell_S : W \to \mathbb{Z}_{\geq 0}$ the length.

W.l.o.g. $\ell(w) \geq \ell(w') = n$. We show the statement by

induction on $n$.

If $q=0$, $w'=e$ and $Bw'B = B \neq BwB$.

If $q \geq 1$, choose $s \in S$, s.t., $\ell(sw') = q-1$.

Hence $\ell(w) > \ell(w's)$ and $w \neq sw'$ and $sw \neq sw'$

and $\ell(sw) \geq \ell(w) - 1 \geq \ell(sw') = q-1$.

By induction, $Bsw'B \neq BwB, BswB$, so

$$Bsw'B \cap BsBBwB = Bsw'B \cap (BwB \cup BswB) = \emptyset$$

which shows

$$Bw'B \cap BwB \subset BsBBsw'B \cap BwB = \emptyset \qquad \square$$

Lecture 12

**Thm B:** $W \supset S$ is a Coxeter group.

**Proof:** We show that $W \supset S$ fulfill the exchange condition (see 1.15 Theorem). Let $w \in W$ with reduced expression $(s_1, \dots, s_n)$ and $s \in S$, s.t., $\ell(sw) \leq \ell(w)$.

By Prop. (3), $BwB \subset BsBwB$, so

$$BsB \subset BwBw^{-1}B \subset \biguplus_{1 \leq i_1 < \dots < i_m \leq n} Bws_{i_m} \cdots s_{i_1} B$$

So $s = ws_{i_m} \cdots s_{i_1}$ for a unique choice $1 \leq i_1 < \dots < i_m \leq n$

and $W = ss_{i_1} \cdots s_{i_m}$, so $m = n-1$. Let $1 \leq j \leq n$, s.t., $\{i_1, \dots, i_n\} \cup \{j\} = \{1, \dots, n\}$. Then

$$ss_1 \cdots s_{j-1} = s_1 \cdots s_j$$ $\square$

## 1.16. Parabolic subgroups and Simplicity

As in the last Section, let $(\mathcal{G}, B, N, S)$ be a Tits system.

Lemma A: (1) Let $w \in W$ with reduced expression $(s_1, \ldots, s_n)$, let $I = \{s_1, \ldots, s_n\}$, then

$$P_I = BW_I B = \langle B, wBw^{-1} \rangle = \langle B, w \rangle$$

(2) $S = \{ w \in W \mid B \cup BwB \subset \mathcal{G} \text{ is a subgroup} \}$ and $S$ is a minimal generating set of $W$

(3) Let $w \in W$, $I, J \subset S$. If $wP_I w^{-1} \subset P_J$, then $w \in P_J$.

Proof: Exercise

Thm A: (a) The map

$$2^S \longrightarrow \{B \subset P \subset G\} \quad I \longmapsto P_I$$

is a bijection.

(b) If $P_J$ is conjugate to $P_I$, $P_I = P_J$

(c) $N_G(P_I) = P_I$

Pf: (a) Surj: Let $B \subset P \subset G$. Then $P$ is a union of double cosets $BwB$. Let $I$ consist of all $s \in S$ that appear in reduced expressions for $w \in P$. Then clearly $P \subset P_I$. Lemma (1) implies $P_I \subset P$.

Inj: Follows from lemma (2)

(b) Since $B, N$ generate $\mathcal{G}$, $P_I$ is conjugate to $P_J$, iff

$wP_Iw^{-1} = P_J$ for some $w \in W$. Then use lemma 3

(c) Same as (b) $\square$

Lemma B: Let $H \trianglelefteq \mathcal{G}$. Then there is a partition $S = I \uplus J$

such that $[I, J] = \{e\}$ and $HB = P_I$.

Proof: Since $H \trianglelefteq \mathcal{G}$, $HB$ is a subgroup.

By Thm. A(a), $HB = P_I$ for some $I \subset S$.

Let $J = S \setminus I$. We have $I = \{s \in S \mid BwB \cap H \neq \emptyset\}$

(Easy Exercise). We need to show $[I, J] = \{e\}$.

Let $s \in I$, $t \in J$. Then $\ell(st) \geq 2$, so

$sBt \subset BstB$ and $tBsBt \subset BtsB \uplus BtstB$

We have $H \cap BsB \neq \emptyset$. Since $H \trianglelefteq G$, we get

$H \cap tBsBt \neq \emptyset$. Hence $\{ts, tst\} \cap W_I \neq \emptyset$.

Now $ts \notin W_I$ since $t \notin W_I$. So

$$tst \in W_I \cap W_{\{s,t\}} = W_{\{s\}} = \{1, s\}.$$

Hence $tst = s \Rightarrow ts = st$ $\square$

We say $W$ is irreducible if there is no partition

$I \uplus J = S$, s.t., $W = W_I \times W_J$

Thm B: Let $W$ be irreducible and assume

(1) $\mathcal{G}$ is generated by conjugates of a solvable $U \trianglelefteq B$

(2) $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$

Then $\mathcal{G}/Z$ is simple (or trivial).

Proof: Let $H \trianglelefteq \mathcal{G}$, $H \not\subseteq Z$. By Lemma B, and irreducibility of $W$, $\mathcal{G} = HB$. Using (1), $\mathcal{G} = HU$.

Then $U/_{U \cap H} \cong HU/_H \cong \mathcal{G}/H$.

Now the left side is solvable and the right side perfect. Hence both are trivial. $\square$

## 1.17 Disecting the unipotent part of a Borel subgroup.

Let $G \supset B \supset T$ be a reductive group with Borel and max. torus.

Let $U = B_u = [B,B]$. Moreover, let $\Delta \subset \Phi^+ \subset \Phi$ be

the corresponding simple, positive and all roots.

Then $\mathfrak{n} = \mathrm{Lie}(U) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$.

For $H \subset U$ $T$-stable, write $\Phi_H = \Phi(H,T) \subset \Phi$, so

$$\mathfrak{h} = \bigoplus_{\alpha \in \Phi_H} \mathfrak{g}_\alpha .$$

Thm: Let $H \subset U$ be a $T$-stable subgroup. Then

$H$ is connected and $\prod_{\alpha \in \Phi_H} U_\alpha \xrightarrow{\sim} H$, where the

product is in any order.

Proof: Assume that $H$ is connected. For $\alpha \in \Phi_H$, we have $U_\alpha \cap H = U_\alpha$ since $\mathrm{Lie}(U_\alpha \cap H) = \mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$. So $U_\alpha \subset H$ and $\pi : \prod_{\alpha \in \Phi_H} U_\alpha \to H$ is well-defined

Case 1: $H$ commutative: Then $\pi$ is a group homomorphism. Moreover $\ker \pi$ is finite and $T$-stable. Hence $T$ acts trivially on $\ker \pi$. But $(\prod U_\alpha)^T = \{e\}$. So $\pi$ is injective. Surjectivity follows from $d\pi$ surjective and $H$ connected

Case 2: $H$ not commutative. In this case $Z = Z(H)^\circ$ has positive dimension and fulfills the assumptions of Case 1. Now, pass to $H/Z$ and apply induction on the dimension.

Now allow non-connected $H$. Then

Let $V = \langle U_\alpha \mid \alpha \notin \Phi_H \rangle$. Then $U = H^\circ V$ and

$H = H^\circ (H \cap V)$. But $H \cap V$ is finite $T$-stable and

hence trivial. $\square$

## Lecture 13

<u>Def.</u>: Let $w = [n] \in W = N_G(T)/T$.

Write $U'_w = U \cap wUw^{-1}$, $U_w = U \cap wU^- w^{-1}$.

Then $\Phi(U'_w, T) = \Phi^+ \cap w(\Phi^+)$, $\Phi(U_w) = \Phi^+ \cap w(\Phi^-)$

and $U = U_w U'_w = U'_w U_w$ $\square$

Remark: (1) In particular

$$w^{-1} \mathcal{U}'_w w \subset \mathcal{U} \qquad \text{and} \qquad w^{-1} \mathcal{U}_w w \subset \mathcal{U}^-$$

(2) Let $\alpha \in \Delta$, and $s = s_\alpha \in S$. Then

$$\Phi^+ \cap s(\Phi^+) = \Phi^+ \setminus \{\alpha\} \qquad \text{and}$$
$$\Phi^+ \cap s(\Phi^-) = \{\alpha\}$$

So we obtain $\mathcal{U} = \mathcal{U}_\alpha \mathcal{U}'_s = \mathcal{U}'_s \mathcal{U}_\alpha$. Since $\mathcal{U}$ is nilpotent and $\mathcal{U}'_s$ has codimension 1 in $\mathcal{U}$, $\mathcal{U}'_s \trianglelefteq \mathcal{U}$.

Since $s(\Phi^+ \cap s(\Phi^+)) = \Phi^+ \cap s(\Phi^+)$, $\mathcal{U}_s$ is normalized by $s$

ence, $Z_\alpha = \langle s, \mathcal{U}_\alpha, T \rangle \subset N_G(\mathcal{U}'_s)$ $\square$

## 1.18 Parametrizing B-double cosets

**Theorem:** Let $w = [n] \in W = N_G(T)/T$. Then, multiplication yields an isomorphism

$$U_w \times \{n\} \times T \times U \xrightarrow{\sim} BwB.$$

**Proof:** We know that the multiplication maps

$U \times T \to B$, $T \times U \to B$, and $U_w \times U'_w \to U$

are isomorphisms of varieties by the structure of solvable group and the previous Theorem.

We can hence write $x \in BwB$ as

$$\underbrace{u}_{U_w}\,\underbrace{u'}_{U'_w}\,\underbrace{t}_{T}\; n\; \underbrace{\underline{t}}_{T}\,\underbrace{\underline{u}}_{U} = \underbrace{u}_{U_w}\; n\; \underbrace{t^n\,\underline{t}}_{T}\;\underbrace{\underbrace{(u')^{t n \underline{t}}}_{(U'_w)^w \subset U}\;\underbrace{\underline{u}}_{U}}_{U}$$

This shows the surjectivity. Now, let

$u\, n = n\, t\, \underline{u}$, then $n^{-1} u\, n = t\, \underline{u} \in U^{-} \cap B = \{e\}$

since $B^{-} \cap B = T$ by 1.7 Corollary (f).

This shows injectivity $\square$

Corollary: Multiplication yields an isomorphism

$$U_w \longrightarrow BwB/B \subset G/B, \quad u \mapsto uwB/B$$

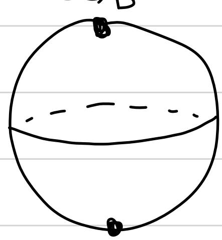
Example: let $G = SL_2$, then $W = \{1, s\}$ and

$BsB = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; c \neq 0 \right\}$, $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}$, and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & ac^{-1} \\ & 1 \end{vmatrix}}_{U_s = U} \underbrace{\begin{vmatrix} & -1 \\ 1 & \end{vmatrix}}_{n} \underbrace{\begin{vmatrix} c & \\ & c^{-1} \end{vmatrix}}_{T} \underbrace{\begin{vmatrix} 1 & -d \\ & 1 \end{vmatrix}}_{U}$$

$sB/B$

$\mathbb{A}^1$ $\Big[$ $\{(a:1) \mid a \in \mathbb{A}^1\}$ $\Big]$ $BsB/B \cong U \cong \mathbb{A}^1$



pt $[$ $\{(1:0)\}$ $\quad B/B \quad$ $]$ $B/B \cong \mathrm{pt}$

In particular, for any $x \in \mathcal{B}$,

$$\mathcal{B} = \mathcal{U}s \cdot x \uplus \mathcal{U} \cdot x \qquad \square$$

<u>Remark</u>: By general results on root systems

$$|\Phi^+ \cap w(\Phi^-)| = \ell(w)$$

In particular, if $(s_1, \ldots, s_n)$ is a reduced expression of $w$, and $\alpha_i$ the simple root corresponding to $s_i$, we have

$$\Phi^+ \cap w\Phi^- = \{\alpha_1, w_{(1)}\alpha_2, \ldots, w_{(n)}\alpha_n\},$$

where $w_{(i)} = s_i \cdots s_n$.

So $\dim BwB/B = \dim U_w = \ell(w) \qquad \square$

## 1.19 Reductive groups yield Tits systems

<u>Thm</u>: Let $G \supset B \supset T$ be reductive with Borel and max. torus. Then $(G, B, N, S)$ forms a Tits system, where $N = N_G(T)$ and $S$ is the choice of simple reflections associated to $B$.

<u>Proof</u>: (1) $B, N$ generate $G$ by 1.14 Lemma.

Also $B \cap N = T \trianglelefteq N$ //(1)

(2) $S$ generates $W$ by std. fact on root systems //(2)

(3) Let $w \in W$, $s = s_\alpha \in S$, $Z = Z_\alpha$, and $B' = wBw^{-1}$.

~~We~~ We ~~nee~~ need to show

$$sBw \subset BswB \cup BwB .$$

For this, consider the $Z_\alpha$-orbit of $wB/B$

$$\begin{tikzcd} Z_\alpha \arrow[rr] \arrow[dr] & & Z_\alpha wB/B \subset G/B \\ & Z_\alpha / wBw^{-1} \cap Z_\alpha = \mathbb{B}_\alpha \arrow[ur, "\sim"'] & \end{tikzcd}$$

By Example 1.18, the orbit decomposes as

$$Z_\alpha wB/B = U_\alpha wB/B \uplus U_\alpha s w B/B .$$

Moreover, we have $Z_\alpha U'_s = U'_s Z_\alpha$ by 1.17 Remark.

Hence

$$\begin{aligned} sBwB \subset Z_\alpha BwB &= Z_\alpha U'_s U_\alpha wB = U'_s Z_\alpha U_\alpha wB \\ &= U'_s Z_\alpha wB = U'_s U_\alpha wB \cup U_\alpha s wB \\ &= BwB \cup BwsB . \end{aligned}$$

//3

(4) Clear, since $\mathbb{B}^T$ is a $W$ torsor //4 $\square$

# Lecture 14

<u>Corollary</u>: (1) If $G$ is almost simple, then $G/Z$ is simple as an abstract group.

For example $PGL_n(k)$ is simple.

(2) We obtain a Bruhat decomposition of the group

$$G = \biguplus_{w \in W} BwB$$

and the flag variety

$$G/B = \biguplus BwB/B$$

The subsets $BwB/B \subset G/B$ are called Bruhat cells and $\mathbb{A}^{\ell(w)} \cong U_w \xrightarrow{\sim} BwB/B$

(3) After choosing representatives $\dot{w} \in N_G(T)$ for $w \in W$, every element $g \in G$ can be uniquely written as

$$g = \underset{\in U_w}{u} \; \dot{w} \; \underset{\in T}{t} \; \underset{\in U}{\underline{u}}$$

## 1.20 Bott-Samelson Varieties

Let $G \supset B \supset T$ be a reductive group with Borel $B$ and max. torus. Let $S \subset W$ denote the set of simple reflections.

For $s \in S$, we denote the associated parabolic subgroup

$$B \subset P_s = P_{\{s\}} = B \cup BsB \subset G.$$

In particular $P_s/B \cong L_s/B_{L_s} \cong \mathbb{P}^1$, where

$L_s = P_s/R_u(P_s)$, $B_{L_s} = B/R_u(P_s)$

Remark: Let $H$ act on $X, Y$ freely from the right and left, respectively. Then $H$ acts on $X \times Y$ from the left via $h(x,y) = (xh^{-1}, hy)$ and the balanced product is the quotient

$$X \overset{H}{\times} Y = X \times Y / H$$

So $[xh, y] = [x, hy] \in X \overset{H}{\times} Y$.

The map $X \overset{H}{\times} Y \to X/H$ is a fibre bundle with fibres $Y/H$ $\square$

**Def:** Let $\underline{x} = (s_1, \dots, s_n) \in S^n$. Then

$$BS(\underline{x}) = P_{s_1} \overset{B}{\times} P_{s_2} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_n} / B$$

is called the <u>Bott-Samelson (Demazure-Hansen)</u> variety associated to $\underline{x}$. Multiplication yields a map:

$$\pi_{\underline{x}} \; BS(\underline{x}) \longrightarrow G/B, \; [p_1, \dots, p_n] \mapsto p_1 \cdots p_n B/B.$$

For $\underline{\varepsilon} \in \{0,1\}^n$, write

$$BS(\underline{x})_{\underline{\varepsilon}} = Bs_1^{\varepsilon_1} B \overset{B}{\times} \cdots \overset{B}{\times} Bs_n^{\varepsilon_n} B/B \subset BS(\underline{x}). \; \square$$

Lemma A (1) The map

$$\begin{array}{c} BS(\underline{x}s) = P_{s_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_n} \overset{B}{\times} P_s / B \\ \downarrow \\ BS(\underline{x}) = P_{s_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_n} / B \end{array}$$

is a $\mathbb{P}^1$-bundle.

(1) We have a decomposition into locally closed subsets

$$BS(\underline{x}) = \biguplus_{\underline{\varepsilon} \in \{0,1\}^n} BS(\underline{x})_{\underline{\varepsilon}}$$

(2) For $\underline{\varepsilon}, \underline{\varepsilon}' \in 2^n$,

$$BS(\underline{x})_{\underline{\varepsilon}} \subset \overline{BS(\underline{x})_{\underline{\varepsilon}'}} \Leftrightarrow \operatorname{supp} \underline{\varepsilon} \subset \operatorname{supp} \underline{\varepsilon}'$$

Lemma B: Let $s \in S$ and $w, x, y \in \mathcal{B}$.

(1) If $\ell(sw) = \ell(w) + 1$, then mult. yields an iso.

$$BsB \overset{B}{\times} BwB \xrightarrow{\sim} BswB \qquad \text{and}$$

$$P_s \overset{B}{\times} BwB \longrightarrow BswB \uplus BwB$$

(2) If $\ell(xy) = \ell(x) + \ell(y)$, then mult. yields an iso.

$$BxB \overset{B}{\times} ByB \xrightarrow{\sim} BxyB$$

Proof: Exercise.

Thm: Let $w \in W$ and $\underline{w} = (s_1, -, s_n)$ a reduced expression and $\pi_{\underline{w}} : BS(\underline{w}) \to G/B$. Then

(1) $\pi_{\underline{w}}$ restricts to an isomorphism

$$\pi_{\underline{w}} : BS(\underline{w})_{\underline{1}^n} \longrightarrow BwB/B$$

(2) $$\operatorname{im} \pi_{\underline{w}} = \overline{BwB/B} = \bigcup_{x \leq w} BxB/B$$

Proof: (1) Follows from Lemma B (1) inductively.

(2) By Lemma A (1), $BS(\underline{w})$ is an iterated $\mathbb{P}^1$-bundle and hence complete. Hence the image is closed. Since $BS(\underline{w}) = \overline{BS_{\underline{1}^n}(\underline{w})}$, by (1), $\operatorname{im} \pi_w = \overline{BwB/B}$.

Moreover, if $x \leq w$, then there is an $\underline{\varepsilon} \in \{0,1\}^n$,

s.t. $x = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}$. Then

$$BxB/B \subset \pi_{\underline{w}}(BS(\underline{x})_{\underline{\varepsilon}}) \qquad \square$$

Lecture 15

**Corollary**: $G/B = \biguplus BwB/B$ is

an affine stratification, that is:

(1) Each stratum $BwB/B$ is locally closed

(2) Each stratum $BwB/B$ is isomorphic to an affine space

(3) The closure of a stratum is a union of strata.

Example: $G = SL_3$, $S = \{s,t\} \subset W$.

$$\begin{tikzcd}[column sep=small]
& sts \arrow[dl, dash] \arrow[dr, dash] & & & & & \mathbb{A}^3 \arrow[dl, dash] \arrow[dr, dash] & \\
ts \arrow[d, dash] \arrow[drr, dash] & & st \arrow[d, dash] \arrow[dll, dash] & \rightsquigarrow & & \mathbb{A}^2 \arrow[d, dash] \arrow[drr, dash] & & \mathbb{A}^2 \arrow[d, dash] \arrow[dll, dash] \\
t \arrow[dr, dash] & & s \arrow[dl, dash] & & & \mathbb{A}^1 \arrow[dr, dash] & & \mathbb{A}^1 \arrow[dl, dash] \\
& e & & & & & \mathbb{A}^0 &
\end{tikzcd}$$

Outlook: let $X = G/B$. If $k = \mathbb{C}$,

(1) $X^{an}(\mathbb{C})$ is a complex manifold and we obtain

$$H^{2n}_{sing}(X^{an}(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{\#\{w \mid \ell(w) = n\}}$$

$$H^{2n+1}_{sing}(X^{an}(\mathbb{C}), \mathbb{Z}) = 0$$

So $\dim H^{\bullet}_{sing}(X^{an}(\mathbb{C}), \mathbb{Z}) = \chi(X) = |W|$

In fact, one can construct an action on $W$ on $H = H^{\bullet}(X^{an}(\mathbb{C}), \mathbb{Q})$, s.t., $H$ is the regular representation of $W$

(2) The varieties $\overline{BwB/B}$ are called <u>Schubert varieties</u> and can be highly singular. $BS(\underline{w})$ provides a desingularization.

## 1.21. Linearly reductive groups

<u>Definition</u>: A linear algebraic group $G$ is <u>linearly reductive</u> if every algebraic rep. $V$ of $G$ is <u>completely reducible</u>, that is, decomposes as a direct sum of irreducibles $\square$

<u>Lemma</u>: Let $G$ be a connected lin. alg. grp. with Lie algebra $\mathfrak{g}$. Assume that every f.d. rep. of $\mathfrak{g}$ is completely reducible. Then $G$ is linearly reductive.

<u>Proof</u>: Let $V$ be an alg. rep. of $G$.

Recall from Corollary 0.4, that $W \subset V$ is $G$-stable iff it is $\mathfrak{g}$-stable.

Hence, a direct sum decomposition in irr. $\mathfrak{g}$-rep's yields a decomposition in irr. $G$-rep's. $\square$

⚠ This relies on char $k = 0$.

<u>Thm</u>: Every f.d. rep. of a reductive Lie alg. is completely reducible. $\square$

<u>Cor</u>: $G$ reductive $\Longrightarrow$ $G$ linearly reductive $\square$

We will prove the Theorem in the next section

Later:

Peter-Weyl $\mathcal{O}(G) = \bigoplus L(\lambda) \otimes L(\lambda)^*$

Borel-Weil $H^0(G/B, \mathcal{L}_\lambda) = L(\lambda)$

$\vdots$

## 1.22. Complete reducibility for reductive Lie algebras

**Def**: Let $\mathfrak{g}$ be a Lie algebra over $k$

(1) $U(\mathfrak{g}) = T^\bullet \mathfrak{g} / (X \otimes Y - Y \otimes X = [X, Y])$

is the universal enveloping algebra of $\mathfrak{g}$.

(3) A symmetric form $B: \mathfrak{g} \otimes \mathfrak{g} \to k$ is called $\mathfrak{g}$-invariant if

$$B([x,y], z) = B(x, [y,z])$$ □

**Lemma**: Let $B: \mathfrak{g} \otimes \mathfrak{g} \to k$ be a symmetric, non-deg. $\mathfrak{g}$-invariant form. Let $\{x_i\}, \{x^i\} \subset \mathfrak{g}$ be dual bases. Then $C_B = \sum x_i x^i \in U(\mathfrak{g})$ does not depend on the choice of basis and is central in $\mathcal{U}(\mathfrak{g})$. $C_B$ is called the <u>Casimir operator</u> associated to $B$.

<u>Proof</u>: $B$ induces an iso. $\mathfrak{g} \to \mathfrak{g}^*$ and a map of $\mathfrak{g}$-rep's:

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^* = \mathrm{End}(\mathfrak{g})$$
$$C_B \longmapsto \mathrm{Id}_{\mathfrak{g}}$$

So $C_B$ does not depend on a choice and commutes with $\mathfrak{g}$-action (since $\mathrm{Id}_{\mathfrak{g}}$ does) $\square$

Example: (1) Let $\mathfrak{g} = \mathfrak{sl}_2$, $B(X,Y) = tr(XY)$, then we get dual bases $\{H, E, F\}$, $\{\frac{1}{2}H, F, E\}$ and

$$C = C_B = \frac{1}{2}H^2 + EF + FE$$

(2) The Killing form $K(X,Y) = tr(ad(X)\, ad(Y))$ is symmetric bilinear. In fact

$$\mathfrak{g} \text{ semisimple} \iff K \text{ non-deg.}$$

If $\mathfrak{g}$ is semisimple, $I \subset \mathfrak{g}$ an ideal, then $I^\perp \subset \mathfrak{g}$ is also an ideal and $\mathfrak{g} = I \oplus I^\perp$ $\square$

From now on, let $\mathfrak{g}$ be semisimple.

<u>Lemma B</u>: Let $V \neq \text{triv}$ be an irred. rep. of $\mathfrak{g}$.

Then, there is a $C_V \in Z(U(\mathfrak{g}))$, s.t., $C$ act by

non-zero scalar on $V$ and $C$ acts by $0$ on $k$

<u>Proof</u>: Let $B_V(X, Y) = \operatorname{tr}(\rho_V(X)\rho_V(Y))$.

<u>Case 1</u>: $B_V$ non. deg.

Let $C_V = C_{B_V}$. By Schur's lemma $\rho_V(C_V) = \lambda \, \mathrm{id}_V$.

and $\operatorname{tr} \rho_V(C_V) = \sum \operatorname{tr}(\rho_V(x_i)\rho_V(x^i)) = \dim \mathfrak{g}$

So $\lambda = \dim \mathfrak{g} / \dim V \neq 0$. // Case 1

<u>Case 2</u>: $B_V$ deg. Exercise

(Hint: Write $\mathfrak{g} = \ker B_V \oplus \mathfrak{g}'$, proceed with

$C_V = C_{B_V|_{\mathfrak{g}'}}$ ) // Case a

$C_V|_k = 0$ is obvious since $k = 0$ $\square$

Lecture 16

Thm. $\mathrm{Ext}^1_{\mathfrak{g}}(k, V) = 0$, so $\mathrm{Hom}_{\mathfrak{g}}(k, -) = (\ )^{\mathfrak{g}}$

is an exact functor.

Proof: Let $0 \to V \to E \to k \to 0$ be an extension.

Assume first that $V$ is irred.

Case 1: $V \neq k$. Pick $C_V$ and $\lambda \neq 0$ as in Lemma B.

Then $E = E_\lambda \oplus E_0 = V \oplus \mathbb{C}$ //

Case 2: $V = k$. Exercise. Use that $\rho_E(\mathfrak{g})$ is nilpotent. //

If $0 \to V_1 \to V \to V_2 \to 0$, then

$$\cdots \to \mathrm{Ext}^1(\mathbb{C}, V_1) \to \mathrm{Ext}^1(\mathbb{C}, V) \to \mathrm{Ext}^1(\mathbb{C}, V_2) \to \cdots$$

So general case follows by induction $\square$

<u>Cor</u>: Let $V$ be a $\mathfrak{g}$-rep. Then

(1) $\mathrm{Hom}_{\mathfrak{g}}(V, -)$ is exact

(2) $\mathrm{Ext}^1_{\mathfrak{g}}(V, W) = 0 \quad \forall\, W$ $\mathfrak{g}$-rep.

Hence, every $\mathfrak{g}$-rep. is completely reducible.

<u>Pf</u>: (1) $\mathrm{Hom}_{\mathfrak{g}}(V, -) = (\mathrm{Hom}_k(V, -))^{\mathfrak{g}}$ and both $\mathrm{Hom}_k(V, -)$ and $(\cdot)^{\mathfrak{g}}$ are exact, see previous Thm.

(2) From (1). $\square$

Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$

where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ Abelian.

Hence, for questions of reducibility, we can assume $\mathfrak{g} = \mathfrak{s}$

This implies Thm 1.21

## 1.23 Induction, Restriction and Frobenius Reciprocity.

Recall: A rational $G$-module $M$ is a possibly infinite-dimensional comodule for $\mathcal{O}(G)$.

We have seen that $M$ is locally finite, that is, $kGm \subset M$ is f.d. for each $m \in M$. We write

$$\underset{\text{f.d.}}{\underset{\uparrow}{\operatorname{Rep} G}} \subset \underset{\text{rational}}{\underset{\uparrow}{G\text{-mod}}}$$

Def: Let $H \subset G$ be alg. grp's.

(1) We denote by

$$\operatorname{Res}^H_G : G\text{-mod} \longrightarrow H\text{-mod}$$

the restriction functor.

(2) For $M \in H$-mod, let $\mathcal{G}$ and $H$ act on $M \otimes \mathcal{O}(\mathcal{G})$ via

$$g(m \otimes f) = m \otimes f(g^{-1}-), \quad h(m \otimes f) = m \otimes f(-h)$$

and call $\mathrm{Ind}_H^{\mathcal{G}}(M) = (M \otimes \mathcal{O}(\mathcal{G}))^H$ the <u>induced representation</u> and

$$\mathrm{Ind}_H^{\mathcal{G}} : H\text{-mod} \to \mathcal{G}\text{-mod}$$

the induction functor. $\square$

Remark: (a) For $M$ f.d. we obtain a chain

$$\operatorname{Hom}(\mathcal{G}, M) \cong \operatorname{Hom}_{Alg}(\underset{\substack{\| \\ \operatorname{Sym} M^*}}{\mathcal{O}(M)}, \mathcal{O}(\mathcal{G}))$$

$$\cong \operatorname{Hom}_k(M^*, \mathcal{O}(\mathcal{G})) \qquad \lambda \mapsto (\lambda(m) \otimes f)$$

$$\cong M \otimes \mathcal{O}(\mathcal{G}) \qquad (m \otimes f)$$

Under this indentification, we obtain

$$\operatorname{Ind}_H^{\mathcal{G}}(M) = \operatorname{Hom}_H(\mathcal{G}, M)$$

$$= \{ f \in \operatorname{Hom}(\mathcal{G}, M) \mid f(gh^{-1}) = h\, f(g) \ \forall g \in \mathcal{G}, h \in H \}$$

(If $M$ is not f.d. this also works, however we have to interpret $M$ as an ind-scheme $M = \bigcup_{\substack{N \subset M \\ \text{f.d.}}} N$ )

(b) The natural map $e^*: \mathcal{O}(G) \to k$, $f \mapsto f(e)$

yields a map

$$\begin{tikzcd}
\mathrm{Hom}(G, M) \arrow[d, "\cong"'] \arrow[dr, "e^*: f \mapsto f(e)"] & \\
M \otimes \mathcal{O}(G) \arrow[r, "\mathrm{id} \otimes e^*"] \arrow[d, equal] & M \\
\mathrm{Ind}_H^G M \arrow[ur, "\varepsilon_M"'] &
\end{tikzcd}$$

In fact $\varepsilon_M$ is a map of $H$-modules. Let

$f \in \mathrm{Hom}_H(G, M)$, then $hf(e) = f(h^{-1}e) = hf(e)$.

Hence, we can write:

$$\varepsilon_M : \mathrm{Res}_G^H \mathrm{Ind}_H^G M \to M .$$

This is functorial in $M$.

(c) If $M \in \mathcal{G}\text{-mod}$, we obtain a natural map

$$\begin{array}{ccc} M & \xrightarrow{m \mapsto (g \mapsto gm)} & \mathrm{Hom}(\mathcal{G}, M) \\ & \searrow_{\text{comodule}} & \wr\wr \\ & & \mathcal{O}(\mathcal{G}) \otimes M \end{array}$$

This map factors via

$$\eta_M : M \longrightarrow \mathrm{Ind}_H^{\mathcal{G}} \mathrm{Res}_{\mathcal{G}}^H M = (\mathcal{O}(\mathcal{G}) \otimes M)^H$$

This is functorial in $M$, too.

**Thm** (Frobenius reciprocity)

Induction and restriction form an adjoint pair

$$\mathrm{Res}^H_G : G\text{-mod} \rightleftarrows H\text{-mod} : \mathrm{Ind}^G_H$$

with unit $\eta$ and counit $\varepsilon$

**Proof**: Exercise: Show that

$$\mathrm{Hom}_G(N, \mathrm{Ind}^G_H M) \rightleftarrows \mathrm{Hom}_H(\mathrm{Res}^H_G N, M)$$

$$\varphi \longmapsto \varepsilon_M \circ \mathrm{Res}^H_G(\varphi)$$

$$\mathrm{Ind}^G_H(\varphi) \circ \eta_M \longleftarrow \varphi$$

are inverse to each other $\square$

**Cor:** For $H' \subset H \subset \mathcal{G}$, we obtain

$$\mathrm{Ind}_H^{\mathcal{G}} \, \mathrm{Ind}_{H'}^{H} M \cong \mathrm{Ind}_{H'}^{\mathcal{G}}$$

**Example:** We have $\mathrm{Ind}_{\{e\}}^{\mathcal{G}} k = \mathcal{O}(\mathcal{G})$, and hence

$$\mathrm{Hom}_{\mathcal{G}}(M, \mathcal{O}(\mathcal{G})) = \mathrm{Hom}_k(M, k) = M^* \qquad \square$$

$$\Big\downarrow \text{Later}$$

$$\mathcal{O}(\mathcal{G}) = \bigoplus_{\lambda \in X(T)_+} L(\lambda) \otimes L(\lambda)^*$$

# Lecture 17

**Thm B** (Tensor Identity) Let $M \in G\text{-mod}$, $N \in H\text{-mod}$.

Then there is a $G$-equivariant isomorphism

$$\mathrm{Ind}_H^G\left(N \otimes \mathrm{Res}^H_G M\right) = \mathrm{Ind}_H^G(N) \otimes M$$

**Proof:** We can write both sides, respectively, as

$$\begin{tikzcd}
\{f \mid f(gh^{-1}) = (h\otimes h)f(g)\} \arrow[d, "\alpha"] \arrow[r, phantom, "\subset"] & \mathrm{Hom}(G, N\otimes M) \\
\{f \mid f(gh^{-1}) = (h\otimes 1)f(g)\} \arrow[ur, phantom, "\subset"] &
\end{tikzcd}$$

Define $\alpha(f)(g) = (1\otimes g)f(g)$. Then

$\alpha$ defines a $G$-equivariant iso. between both sides. $\square$

## 1.24. Induction via global sections

Recall: For $H \subset G$ a closed subgroup in a affine alg. grp., there is a geometric quotient

$$\pi : G \to G/H$$

such that

(1) $\pi$ is a set-theoretical quotient

(2) $\pi$ is open

(3) $(\pi_* \mathcal{O}_G)^H = \mathcal{O}_{G/H}$, so $\mathcal{O}_{G/H}(U) = \mathcal{O}_G(\pi^{-1}(U))^H$

(4) $G/H$ is smooth and quasi-projective

See Section 0.3 $\square$

For $M \in H\text{-mod}$, we can construct a sheaf of $\mathcal{O}_{G/H}$-modules $\mathcal{L}_M$ on $G/H$ via

$$\mathcal{L}_M(U) = \{ f \in \mathrm{Hom}(\pi^{-1}(U), M) \mid f(gh) = h^{-1}f(g) \ \forall h \in H, g \in G \}$$
$$= \mathrm{Hom}_H(\pi^{-1}(U), M) = (M \otimes \mathcal{O}_G(\pi^{-1}(U)))^H.$$

In particular:

$$H^0(G/H, \mathcal{L}_M) = \Gamma(G/H, \mathcal{L}_M) = \mathcal{L}_M(G/H) = \mathrm{Ind}_H^G M$$

One may show that $\mathcal{L}_M$ is a locally free $\mathcal{O}_{G/H}$-module, and hence corresponds to the sections of a vector bundle:

Consider the balanced product / associated bundle

$$p : G \overset{H}{\times} M \to G/H$$

A section of $p$ over $\mathcal{U} \subset G/H$ open, is a map

$$s: \mathcal{U} \to G \overset{H}{\times} M,$$

such that $ps = id_{\mathcal{U}}$. Denote by $\Gamma(\mathcal{U}, G \overset{H}{\times} M)$ the set of sections. Then, there is an isomorphism

$$\begin{aligned} \mathcal{L}_M(\mathcal{U}) &\longrightarrow \Gamma(\mathcal{U}, G \overset{H}{\times} M) \\ f &\longmapsto (x \mapsto [x, f(x)]) \end{aligned}$$

$\cap$

$\mathrm{Hom}(\pi^{-1}(\mathcal{U}), M)$

Example: If $M = k$ is the trivial representation, then $\mathcal{L}_k = \mathcal{O}_{G/H}$ $\square$

## 1.25 Peter-Weyl, first version

<u>Definition:</u> Let $L, M \in \mathfrak{g}$-mod, s.t. $L$ is simple.

Then $M_L \subset M$, the $L$-isotypic component of $M$, is the sum of all submodules in $M$ isomorphic to $L$.

<u>Lemma:</u> Let $M \in \mathfrak{g}$-mod

(a) There is an isomorphism of $\mathfrak{g}$-modules

$$\mathrm{Hom}_{\mathfrak{g}}(L, M) \otimes L \to M_L, \quad f \otimes v \mapsto f(v)$$

(b) If $M$ is a sum of simple rep.,

then it is a direct sum of simple rep. and

$$M = \bigoplus_{L \in \mathrm{Irr}(\mathfrak{g})} M_L = \bigoplus_{L \in \mathrm{Irr}(\mathfrak{g})} \mathrm{Hom}_{\mathfrak{g}}(L, M) \otimes L \qquad \square$$

Proof: Omitted $\square$

Thm (Peter-Weyl) Let $G$ be reductive, then

$$\mathcal{O}(G) \xleftarrow{\sim} \bigoplus_{L \in \mathrm{Irr}(G)} L^* \otimes L$$

$$g \mapsto \lambda(gv) \longleftarrow\!\!\shortmid \lambda \otimes v$$

is an isomorphism of $G \times G$ - modules

Proof: Follows from Lemma and 1.23 Example. $\square$

Example: Let $G = T$ a torus. Then

$\mathrm{Irr}(T) = X(T)$ and

$$\mathcal{O}(T) = k[X(T)] = \bigoplus_{\chi \in X(T)} k\chi = \bigoplus_{\chi \in X(T)} k_{\chi^{-1}} \otimes k_\chi$$

Here $T \times T$ acts on $k\chi$ via

$$(t_1, t_2)(a\chi)(t) = a\chi(t_1^{-1} t t_2)$$

$$= \chi(t_1^{-1})\, a\chi(t)\, \chi(t_2)$$

$$= \chi(t_1^{-1}) \chi(t_2)\, a\chi(t) .$$

So $\quad k\chi \cong k_{\chi^{-1}} \otimes k_{\chi}$

## 1.26 Example: $SL_2$

Consider the map

$$\mathcal{G} = SL_2 \longrightarrow k^2/\{0\} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto Ae_1 = \begin{pmatrix} a \\ c \end{pmatrix}.$$

This is the geometric quotient

$$\pi : SL_2 \to SL_2/U$$

where $U = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \subset B = \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \subset SL_2$.

By the Peter-Weyl thm.

$$\mathrm{Ind}_U^{\mathcal{G}} k = \mathcal{O}(\mathcal{G}/U) = \mathcal{O}(\mathcal{G})^U = \bigoplus_{L \in \mathrm{Irr}(\mathcal{G})} L^* \otimes L^U = \bigoplus_{L \in \mathrm{Irr}(\mathcal{G})} L^*$$

Compare this to

$$\mathcal{O}(k^2 \setminus \{0\}) = k[a,c] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} k[a,b]_n$$

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How to obtain the irreducible constituents?

$$B \longrightarrow B/U = T \cong \mathbb{G}_m.$$

Hence, for each $\lambda \in X(T) = \mathbb{Z}$, we obtain a one-dimensional representation $\lambda : B \to \mathbb{G}_m$.

Denote this by $k_\lambda \in \operatorname{Rep} B$.

Then $$(L \otimes k_\lambda)^B \cong \underset{\substack{\uparrow \\ \text{highest weight}}}{L^U} \cap \underset{\substack{\nwarrow \\ -\lambda \text{ weight space}}}{L_{-\lambda}}$$

$$= \begin{cases} v^+ \otimes k_\lambda, & L = L(-\lambda) \\ 0, & \text{else.} \end{cases}$$

Hence, we obtain Borel-Weil for $SL_2$:

$$\mathrm{Ind}_B^G k_\lambda = H^0(G/B, \mathcal{L}_\lambda)$$

$$= (\mathcal{O}(G) \otimes k_\lambda)^B$$

$$= \bigoplus_{L \in \mathrm{Irr}(G)} L^* \otimes (L \otimes k_\lambda)^B$$

$$= \begin{cases} L(-\lambda)^* & , -\lambda \in \mathbb{Z}_{\geq 0} \\ 0 & , \text{else} \end{cases}$$

Compare this to $G/B \cong \mathbb{P}^1 = \{[a:c]\}$

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = k[a,c]_n$$

So $\mathcal{L}_\lambda = \mathcal{O}(-\lambda)$

Aside: The Borel-Weil-Bott theorem also computes the higher cohomology groups. For $SL_2$, we can use Serre duality

$$\begin{array}{ccc} H^1(G/B, \mathcal{L}_\lambda) & = & H^0(G/B, \mathcal{L}_{-\lambda+2})^* \\ \| & & \| \\ H^1(\mathbb{P}^1, \mathcal{O}(-\lambda)) & = & H^0(\mathbb{P}^1, \mathcal{O}(\lambda-2))^* \end{array}$$

$$\begin{aligned} s\cdot\lambda &= s(\lambda+\rho)-\rho \\ &= s(\lambda+1)-1 \\ &= -\lambda-2 \end{aligned}$$

## 1.27. Parabolic Induction

Let $G \supset B \supset T$ be a reductive group with Borel and max. torus. Let $U = R_u(B)$. So, we obtain maps

$$T \cong B/U \xleftarrow{\pi} B \longrightarrow G$$

For $\lambda \in X(T)$, denote by $k_\lambda \in \mathrm{Rep}(T)$ the one-dim. rep. By abuse of notation, denote by $k_\lambda \in \mathrm{Rep}(B)$ inflation to $B$ via $\pi$ (so $u k_\lambda = 0 \ \forall u \in U$)

From now, we abbreviate the <u>parabolically</u> <u>induced</u> rep.

$$H^0(\lambda) = \mathrm{Ind}_B^G k_\lambda = \{ f : \mathrm{Hom}(G, k) \mid f(gb) = b \cdot f(g) \}$$

$$= \{ f \in \mathcal{O}(G) \mid f(gb) = \lambda^{-1}(b) f(g) \}$$

Remark A: In general, consider

$$P/R_uP = L \leftarrow P \to G$$

Then for $V \in Rep(L)$, we get $Ind_P^G V \in Rep(G)$ $\square$

Proposition: Let $V \in G$-mod

(1) $Hom_T(k_\lambda, V) = V_\lambda$, $Hom_B(k_\lambda, V) = V_\lambda \cap V^U$

(2) If $V \neq 0$, $V^U \neq 0$.

(3) If $V \neq 0$, there is a $\lambda \in X(T)$, s.t. $Hom(k_\lambda, V) \neq 0$.

(4) If $V \neq 0$ and $\dim V < \infty$, there is a $\lambda \in X(T)$, s.t.

$$Hom_B(V, k_\lambda) \neq 0$$

(5) $V^G = V^B$ $\square$

**Proof:** (1) First statement is clear, second uses that $B = TU$

(2) By Lie-Kolchin (0.7 Thm.) $\mathbb{P}(V)^U \neq \emptyset$,

so $U$ has a common eigenvector in $V$. Since $U$ is

unipotent the eigenvalue is zero

(3) Since $T$ normalizes $U$, $V^U$ is a $T$-rep. and

hence decomposes in weight spaces (see 1.4).

Take one $\lambda$ with $(V^U)_\lambda \neq 0$

(4) For $V$ finite-dimensional

$\mathrm{Hom}_B(V, k_\lambda) = \mathrm{Hom}_B(k_\lambda^*, V^*) = {}_B\mathrm{Hom}(k_{-\lambda}, V^U)$.

Now apply (3)

(5) $V^B = \mathrm{Hom}_B(\mathrm{Res}^B_G k, M) \overset{FR}{=} \mathrm{Hom}_G(k, \mathrm{Ind}^G_B M)$

$\overset{TI}{=} \mathrm{Hom}_G(k, (\mathrm{Ind}^G_B k) \otimes M) = \mathrm{Hom}_G(k, M) = V^G$

where we use that $\mathrm{Ind}^G_B k = H^0(G/B, \mathcal{O}_{G/B}) = k$, since

$G/B$ is projective $\square$

Remark B: (5) yields the surprising conclusion, that

$\mathrm{Res}^B_G$ is fully faithful: Let $V, W \in \mathrm{Rep}(G)$, then

$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G = \mathrm{Hom}(V, W)^B = \mathrm{Hom}_B(V, W)$

Corollary: Let $0 \neq V \in \mathrm{Rep}(G)$. Then there is a $\lambda \in X(T)$, s.t.,

$$\mathrm{Hom}_G(V, H^0(\lambda)) \neq 0$$

Proof: By Frobenius reciprocity (1.23 Thm A)

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G k_\lambda) = \mathrm{Hom}_B(V, k_\lambda)$$

Now use Prop. (4) $\square$

Hence, every simple is a submodule of some $H^0(\lambda)$.

We did not use semisimplicity yet.

## 1.28 Highest weights and simplicity of induced rep.'s

Next, we will analyse weigth spaces.

Def: (1) We denote $R(T) = \mathbb{Z}[X(T)]$ and for $\lambda \in X(T)$ the corresponding element in $R(T)$ by $e^\lambda$. For $V \in \mathrm{Rep}(T)$,

$$\mathrm{ch}(V) = \sum_{\lambda \in X(T)} \dim V_\lambda e^\lambda$$

is called the <u>character of V</u>.

(2) Let $\lambda \in X(T)$. For a choice of positive and simple roots $X(T) \supset \Phi \supset \Phi^+ \supset \Delta$, write

$$\lambda \leq \mu \iff \mu - \lambda \in \mathbb{N}\Delta = \mathbb{N}\Phi^+.$$

Moreover, the set of <u>dominant weights</u> is

$$X(T)_+ = \{ \lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha^\vee \in \Delta \}$$

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<u>Proposition</u> : Let $V \in \operatorname{Rep} \mathcal{G}$.

(1) Let $w = [n] \in W = N_{\mathcal{G}}(T)/T$. Then $n(V_\lambda) = V_{w(\lambda)}$

and $\dim V_\lambda = \dim V_{w(\lambda)}$ for all $\lambda \in X(T)$

(2) $\operatorname{ch}(V) \in R_T^W$

(3) If $\lambda \in X(T)$ is maximal amongst the weights of $V$,

then $V_\lambda \subset V^U = V^{\operatorname{Lie} U}$

<u>Proof</u>: (1) + (2) Clear by definition.

(3) For $\alpha \in \Phi^+$, $u_\alpha = \operatorname{Lie} U_\alpha$ maps $V_\lambda$ to $V_{\lambda+\alpha} = 0$,

since $\lambda \leq \lambda + \alpha$ $\square$

<u>Remark</u>: For an Abelian category $\mathcal{A}$, denote

$$K_0(\mathcal{A}) = \mathbb{Z}\left(\mathrm{Ob}\,\mathcal{A}/\cong\right) \Big/ \left\langle [B] = [A] + [C], \text{ if there is a s.e.s. } A \to B \to C \right\rangle$$

Then

$$\begin{tikzcd}
K_0(\mathrm{Rep}(G)) \arrow[r, "\mathrm{Res}_G^T"] \arrow[d] & K_0(\mathrm{Rep}\,T) \arrow[d, "\wr"] & {[V]} \arrow[d, maps to] \\
R_T^W \arrow[r, hook] & R_T & \mathrm{ch}(V)
\end{tikzcd}$$

We fix the choice of $\Phi \supset \Phi^+ \supset \Delta$ corresponding to $B$.

Recall that there is a unique $w_0 \in W$, s.t., $w_0(\Phi^+) = -\Phi^+$

and $B^- = B^{w_0}$, $U^- = U^{w_0}$. Moreover $U^- B = B^- B \subset G$

is dense, since $B^- B = w_0 B w_0 B$ and $B w_0 B$ is dense.

<u>Lemma:</u> Let $\lambda \in X(T)$, s.t., $H^0(\lambda) \neq 0$

(1) $\dim \left( H^0(\lambda)^{U^-} \right) = 1$ and $H^0(\lambda)^{U^-} = H^0(\lambda)_\lambda$

(2) $\dim H^0(\lambda)_{w_0(\lambda)} = 1$ and $H^0(\lambda)^{U^+} = H^0(\lambda)_{w_0(\lambda)}$

(3) All weights $\mu$ in $H^0(\lambda)$ are of the form

$$\lambda \leq \mu \leq w_0 \lambda$$

Proof: (1)(3) For $f \in H^0(\lambda)^{U^-} \subset \mathcal{O}(G)$, we get

$$f(u^- t u) = \lambda(t)^{-1} f(1) \quad \text{for all } u^- \in U,\ u \in U,\ t \in T.$$

Hence we obtain that the composition of maps

$$\begin{tikzcd}
H^0(\lambda)^U \subset \mathcal{O}(G) \arrow[r, phantom, "\overset{\text{dense}}{\subset}"] \arrow[drr, bend right, "\varepsilon_{k\lambda}|_{H^0(\lambda)^U}"'] & & \mathcal{O}(B^-B) \arrow[d] \\
& & k_\lambda
\end{tikzcd}$$

is injective. So $\dim H^0(\lambda)^{U^-} \leq 1$ and $H^0(\lambda)^{U^-} \subset H^0(\lambda)_\lambda$.

Since $H^0(\lambda) \neq 0$ by assumption $H^0(\lambda)^{U^-} \neq 0$ by 1.27 Prop (2).

So $\dim H^0(\lambda)^{U^-} = 1$. If $\mu$ is a minimal weight in $H^0(\lambda)$, then $H^0(\lambda)^{U^-} \subset H^0(\lambda)_\mu$ by Prop (3) (Reverse the roles of $U, U^-$).

Hence $H^0(\lambda)^{U^-} = H^0(\lambda)_\lambda$ and $\lambda \leq \mu$ if $\mu$ is a weight of $H^0(\lambda)$. Now $w_0$ permutes weights in $H^0(\lambda)$, so $\lambda \leq w_0\mu \Rightarrow \mu \leq w_0\lambda$

(2) Use $V^U = V^{w_0 U^- w_0} = w_0(V^{U^-})$ and (1) $\square$

<u>Thm</u>: (1) If $H^0(\lambda) \neq 0$, then $H^0(\lambda)$ is simple.

(2) For all $L \in \operatorname{Irr}(G)$, there is a $\lambda \in X(T)$ with

$L \cong H^0(\lambda)$ and $w_0(\lambda)$ is the highest weight of $L$.

<u>Pf</u>: (1) If $H^0(\lambda)$ is not simple, write $H^0(\lambda) = V \oplus W$ for $V, W \neq 0$. Then $V^U, W^U \neq 0 \Rightarrow \dim H^0(\lambda)^U \geq 0$ ↯ $\square$

(2) Follows from 1.27 Cor. $\square$

If $H^0(w_0\lambda) \neq 0$, we denote $L(\lambda) = H^0(w_0\lambda)$

the corresponding simple module of highest weight $\lambda$.

## 1.29. The Borel-Weil theorem

**Lemma**: Let $\lambda \in X(T)$, $s \in S$ and $\alpha = \alpha_s \in \Delta$. Then

$$H^0(P_s/B, \mathcal{L}_\lambda) \neq 0 \iff \langle -\lambda, \alpha^\vee \rangle \geq 0 .$$

In this case, there is an $f \in H^0(P_s/B, \mathcal{L}_\lambda)$, s.t.

$$f(U_{-\alpha} B/B) = \text{const} \neq 0$$

**Proof**: We have $P_s/B \cong \mathbb{P}^1$. Under this iso.

$\mathcal{L}_\lambda$ corresponds to $\mathcal{O}(-\langle \lambda, \alpha^\vee \rangle)$.

The statement follows from a computation for $\mathbb{P}^1$

(Exercise), see also Section 1.26. $\square$

**Thm:** Let $\lambda \in X(T)$. Then $H^0(G/B, \mathcal{L}_\lambda) \neq 0 \iff$

$-\lambda$ antidominant.

**Pf:** "$\Rightarrow$" Follows from lemma

"$\Leftarrow$" Let $s \in S$ and $\alpha = \alpha_s$. Denote

$$U_s^- = U^- \cap sU^-s \xleftarrow{\sim} \prod_{\alpha \in \Phi - \{-\alpha\}} U_\alpha$$

Then, we obtain a diagram

$$\begin{tikzcd}
U_s^- \times P_s/B \arrow[r, "\sim", "m"'] \arrow[dr, "\pi_2"'] & B^- P_s/B \arrow[d, shift right, "p_s"'] \\
& P_s/B \arrow[u, shift right, "i_s"']
\end{tikzcd}$$

- $m$ is given by multiplication and an iso. using
- $i_s$ is the inclusion

- $p_s$ is defined by $\pi_2 = p_s m$, $\pi_2$ the projection

Then $p_s i_s = id$ and we obtain maps

$$p_s^*: H^0(P_s/B, \mathcal{L}_\lambda) \rightleftarrows H^0(B^- P_s/B, \mathcal{L}_\lambda) : i_s^*$$

$$f \longmapsto (u^- pB/B \mapsto f(pB/B))$$

$$f|_{P_s/B} \longleftarrow\!\shortmid f$$

such that $i_s^* p_s^* = id$, so $i_s^*$ is surjective.

Now, pick $0 \neq f_s \in H^0(P_s/B, \mathcal{L}_\lambda)$, s.t.

$f_s(B/B) = f_t(B/B)$ $\forall s,t \in S$ and

$f_s|_{U_{-\alpha_s}B/B} = \text{const}$. This is possible by

the previous lemma and the assumption.

Let $g_s = p_s^*(f_s)$. Then, by construction

$g_s = g_t$ restricted to $B^- P_s/B \cap B^- P_s/B \simeq B^- B/B$.

(Exercise : Show this)

Hence, $(g_s)_{s\in S}$ glues to a function $g$ on

$$W = \bigcup_{s\in S} B^- P_s/B.$$

By the Bruhat-decomposition (Exercise)

$W$ has codimension 2. Since $G/B$ is normal,

$g$ extends to $G/B$. So $0 \neq g \in H^0(\lambda)$ $\square$

**Corollary A** (Borel-Weil) Let $\lambda \in X(T)$, s.t. $-\lambda$ is dominant. Then $H^0(\lambda) = L(w_0\lambda) \neq 0$ is the unique simple representation with highest weight $w_0\lambda$ $\square$

**Corollary B**: If $L$ is simple and $\lambda$ the highest weight of $L$, then $L = L(\lambda) = H^0(w_0\lambda)$ $\square$

**Corollary C**: $L(\lambda)^* = L(-w_0\lambda)$

**Corollary D**: $\mathcal{O}(g) \cong \bigoplus_{\lambda \in X(T)_+} L(\lambda)^* \otimes L(\lambda)$

**Corollary E**: $K_0(\mathrm{Rep}(g)) \to K_0(\mathrm{Rep}(T))$ is inj.

## 1.30 Work of Demazure

In the following, we will discuss the proof of the Borel-Weil-Bott thm. and character formula due to Demazure:

- A Very Simple Proof of Bott's Theorem
- Une nouvelle formule des caractères ⚠ INCORRECT
  ↓
- Andersen: Schubert varieties and Demazure's character formula

Demazure's work is based on:

- The fibration $\underset{\underset{\mathbb{P}^1}{\cong}}{P_s/B} \to G/B \to G/P_s$

- The geometry of Bott-Samelson resolutions

$$BS(\underline{w}) \to \overline{X_w}$$

of Schubert varieties, see 1.20.

- Tools from cohomology of coherent sheaves

(Leray spectral sequence, ...) $\rightsquigarrow$ 1.??

⚠ Warning ⚠

The work of Demazure (and related publications) uses a different convention for simple/positive roots. Namely $\Delta \subset \Phi^+$ are now the roots associated to $B^-$. From now, we use this convention.

This has the following consequences:

$$H^0(G/B, \mathcal{L}_\lambda) = \begin{cases} L(\lambda) & \text{if } \lambda \text{ is dominant} \\ 0 & \text{else} \end{cases}$$

Here $L(\lambda)$ is the simple module of highest weight $\lambda$ and $\mathfrak{n} \subset \mathfrak{b}$ decreases the weight.

Moreover, we will sometimes assume that $G$ is <u>simply connected</u>, that is, $Y(T) = \mathbb{Z}\Phi^\vee$. This is irrelevant for the geometry of $G/B$: For $G$ reductive, there is a $G_{ss} \to G$, s.t. $G_{ss}/B_{ss} \xrightarrow{\sim} G/B$.

## 1.31 Some facts on sheaf cohomology.

References:

- Hartshorne, Stacks Projects, EGA, ...

For an $\mathcal{O}_X$-module $M$ on a variety $X$, we can consider its global sections

$$H^0(X, M) = \Gamma(X, M) = M(X).$$

This can be made relative and derived.

For a morphism $f: X \to Y$ of varieties, there adjoint functors

$$f_*: \mathcal{O}_X\text{-mod} \leftrightarrows \mathcal{O}_Y\text{-mod} : f^*$$

They, amongst many other things, fulfill

(1) Adjunction:

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_* M, N) = \mathrm{Hom}_{\mathcal{O}_X}(M, f^* N)$$

(2) Exactness + derived

For a s.e.s. $0 \to A \to B \to C \to 0$ in $\mathrm{Coh}(X)$,

there is a functorial l.e.s.

$$0 \to f_* A \to f_* B \to f_* C \to R^1 f_* A \to R^1 f_* B \to R^1 f_* B$$

$$\to R^2 f_* A \to \dots$$

Here, $R^n f_* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$ is the n-th

derived functor of $f_*$

(3) Global sections:

If $Y = pt = \mathrm{Spec}\, k$, $R^i f_* M = H^i(X, M)$

(4) Leray spectral sequence

For a fibration

$$F \to E \xrightarrow{\pi} X$$

such that "$M$ is constant along $X$"

$R^n \pi_* M$ is locally of the form $H^n(F, M)$,

and there is the Leray spectral sequence

$$H^p(X, H^q(F, M)) \Rightarrow H^{p+q}(E)$$

(5) Projection formula

Let $V \in \mathcal{O}_Y$-mod be finite locally free on $Y$.

(so $V$ is a vector bundle).

Then there is a natural iso

$$R^n f_* M \otimes V \xrightarrow{\sim} R^n f_* (M \otimes f^* V)$$

(6) Vanishing for affine schemes

For $X$ affine an $M$ quasi-coherent,

$$H^n(X, M) = 0 \quad \forall n > 0$$

(7) Dimension vanishing

$$H^n(X, M) = 0 \quad \forall n > \dim X$$

(8) Mayer-Vietories

Consider a open cover

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook] \\ V \arrow[r, hook] & X \end{tikzcd}.$$

Then, there is a l.e.s

$$0 \to f_* M \to f_* M|_U \oplus f_* M|_V \to f_* M|_{U \cap V} \to R^1 f_* M \to \dots$$

<u>Example</u>: (1) Let $G \supset H$ be a lin. alg. group with subgroup $H$. Let $M \in H\text{-mod}$. Then, as in 1.24 we can consider the induced rep.

$$\mathrm{Ind}_H^G M = H^0(G/H, \mathcal{L}_M)$$

One may show that the derived functors agree, i.e.,

$$R^n \mathrm{Ind}_H^G M = H^n(G/H, \mathcal{L}_M)$$

(2) In the above setup, let $M \in \mathrm{Rep}\, G$, $N \in H\text{-mod}$. The (derived) tensor identity 1.23 Thm. B and projection formula agree. For this, first note that ther is an isomorphism of total spaces of vector bundles over $G/H$

$$\mathcal{G} \overset{\mathcal{H}}{\times} M \longrightarrow \mathcal{G}/\mathcal{H} \times M \qquad [g, m] \mapsto (g\mathcal{H}, gm)$$

Hence $\mathcal{L}_M = \mathcal{O}_{\mathcal{G}/\mathcal{H}} \otimes_k M = f^* M$, wher $f : \mathcal{G}/\mathcal{H} \to pt$.

So we obtain

$$\begin{array}{ccc} R^n \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} (N \otimes \operatorname{Res}_{\mathcal{G}}^{\mathcal{H}} M) & \cong & (R^n \operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}} N) \otimes M \\ \| & & \| \end{array}$$

$$H^n(\mathcal{G}/\mathcal{H}, \mathcal{L}_N \otimes \mathcal{L}_M) = R^n f_* (\mathcal{L}_N \otimes f^* M) \cong (R^n f_* \mathcal{L}_N) \otimes M = H^n(\mathcal{G}/\mathcal{H}, \mathcal{L}_N) \otimes M$$

Exercise: Compute $H^i(\mathbb{P}^1, \mathcal{O}(n))$

## 1.32 Dot-action and Borel-Weil-Bott for rank 1

<u>Def:</u> Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{\alpha \in \Delta} \omega_{\alpha^\vee}$

where $\{\omega_\alpha\}$ the dual basis for $\{\alpha^\vee\}$ in $X(T)_{\mathbb{Q}}$

Then, we define the <u>dot-action</u> of $W$ on $X(T)$ via

$$w \cdot \lambda = w(\lambda + \rho) - \rho \qquad \square$$

<u>Remark:</u> For $s \in S$ and $\lambda \in X(T)$

$$s \cdot \lambda = s(\lambda + \rho) - \rho = s(\lambda) + \rho - \langle \rho, \alpha^\vee \rangle \alpha - \rho$$

$$= s(\lambda) - \langle \Sigma \omega_\alpha, \alpha^\vee \rangle \alpha = s(\lambda) - \alpha$$

Since $W$ is generated by simple reflections,

this shows that $w \cdot$ preserves $X(T)$ $\square$

Reminder: (1) For $n \in \mathbb{Z}$, we have

$$H^0(\mathbb{P}^1, \mathcal{O}(n)) = k[X,Y]_n$$

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) = H^0(\mathbb{P}^1, \mathcal{O}(-n-2))^* = k[X,Y]^*_{-n-2}$$

$$H^i(\mathbb{P}^1, \mathcal{O}(n)) = 0 \text{ if } i \neq 0,1 \text{ or } n=-1$$

(2) Let $G = SL_2$, $X(T) \cong \mathbb{Z}$ $\underbrace{\left(\left(\begin{smallmatrix} t & \\ & t^{-1} \end{smallmatrix}\right) \mapsto t\right)}_{= \rho} \mapsto 1$

Then $\mathcal{L}_\lambda$ on $G/B$ corresponds to $\mathcal{O}(\lambda)$ on $\mathbb{P}^1$.

$$H^0(G/B, \mathcal{L}_\lambda) = \begin{cases} L(\lambda) & , \lambda \geq 0 \\ 0 & , \text{else} \end{cases}$$

$$H^1(G/B, \mathcal{L}_\lambda) = \begin{cases} L(\underbrace{-\lambda-2}_{= s\cdot\lambda}) & , -\lambda-2 \geq 0 \\ 0 & , \text{else} \end{cases}$$

(3) let $G \supset B \supset T$ be reductive, $s \in S$, $\alpha = \alpha_s$

$P_s = B \cup BsB$. Then $SL_2 \to P_s$ and

$$\begin{array}{ccccc} \mathcal{O}(\langle \lambda, \alpha^\vee \rangle) & & \mathcal{L}_{\langle \lambda, \alpha^\vee \rangle} & & \mathcal{L}_\lambda \\ | & \cong & | & \cong & | \\ \mathbb{P}^1 & & SL_2/B & & P_s/B \end{array}$$

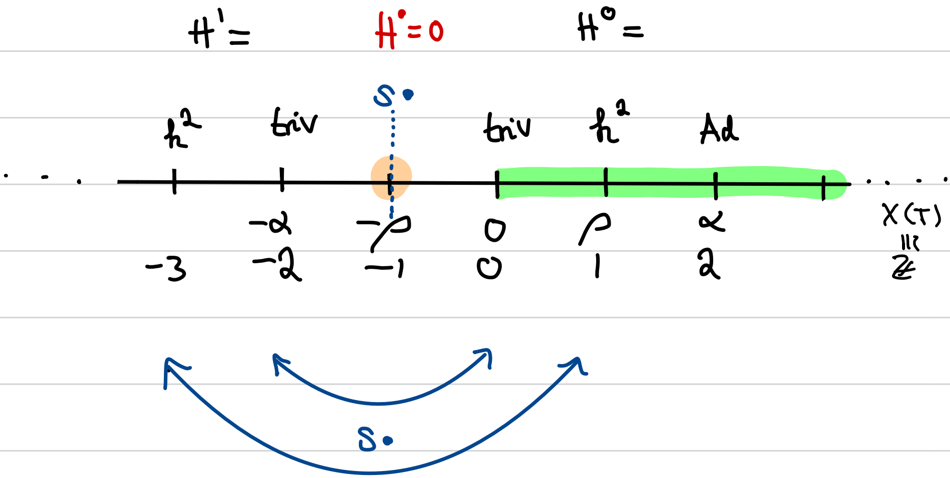
$$ch(H^0(P_s/B, \mathcal{L}_\lambda)) = \begin{cases} e^{s(\lambda)} + \dots + e^{\lambda} & \langle \lambda, \alpha^\vee \rangle \geq 0 \\ 0 & \text{else} \end{cases}$$

$$ch(H^1(P_s/B, \mathcal{L}_\lambda)) = \begin{cases} e^{\lambda + \alpha} + \dots + e^{s \cdot \lambda} & \langle \lambda, \alpha^\vee \rangle \leq -2 \\ 0 & \text{else} \end{cases}$$

$$= ch(H^0(P_s/B, \mathcal{L}_{s \cdot \lambda}))$$

$ch(H^i(P_s/B, \mathcal{L}_\lambda)) = 0$ if $i \neq 0, 1$ or $\langle \lambda, \alpha^\vee \rangle = -1$

Visualisation: For $SL_2$, we can visualise this as



## 1.33 Borel-Weil-Bott after Demazure

M. Demazure - A Very Simple Proof of Bott's Theorem

<u>Lemma</u>: "An Easy Lemma" Let $s \in S$, $V \in \operatorname{Rep} P_s$, $\alpha = \alpha_s$

$\lambda \in X(T)$, s.t. $\langle \lambda, \alpha^\vee \rangle = -1$. Then

$$H^n(G/B, \mathcal{L}_V \otimes \mathcal{L}_\lambda) = 0 \qquad \forall n$$

<u>Proof</u>: Denote $f : P_s/B \to pt = \operatorname{Spec}(k)$. Since

$V$ is a rep. of $P_s$ by 1.30 Example (2) and

1.31 Reminder (3)

$$H^q(P_s/B, \mathcal{L}_V \otimes \mathcal{L}_\lambda) = V \otimes H^q(P_s/B, \mathcal{L}_\lambda) = 0 \quad \forall q$$

The fibration $P_s/B \to G/B \xrightarrow{\pi_s} G/P_s$ yields the

Leray spectral sequence (1.31 Facts)

$$H^p(G/P_s, H^q(P_s/B, \mathcal{L}_V \otimes \mathcal{L}_\lambda)) \Rightarrow H^{p+q}(G/B, \mathcal{L}_V \otimes \mathcal{L}_\lambda)$$

which implies that the r.h.s. vanishes $\square$

<u>Proposition</u>: Let $s \in S$, $\alpha = \alpha_s \in \Delta$ and $\lambda \in X(T)$ with $\langle \lambda, \alpha^\vee \rangle \geq 0$

Then there is a s.e.s. of $B$-modules

$$0 \to K \to H^0(P_s/B, \mathcal{L}_\lambda) \to k_\lambda \to 0$$

where $K = 0$ if $\langle \lambda, \alpha^\vee \rangle = 0$, $K = k_{s(\lambda)}$ if $\langle \lambda, \alpha^\vee \rangle = 1$.

If $\langle \lambda, \alpha^\vee \rangle \geq 2$, there is a s.e.s. of $B$-modules

$$0 \to k_{s(\lambda)} \to K \to H^0(P_s/B, \mathcal{L}_{\lambda-\alpha}) \to 0$$

⚠ These are not s.e.s. of $P_s$-modules!

<u>Proof Sketch</u>: The $\mathfrak{b}$ action on the weight spaces of $H^0(P_s/B, k_\lambda)$ can be visualised as

$$\overbrace{\overbrace{k_{s(\lambda)} \leftarrow \underbrace{k_{s(\lambda)+\alpha} \leftarrow \cdots \leftarrow k_{\lambda-\alpha}}_{H^0(P_s/B,\, k_{\lambda-\alpha})}}^{K} \leftarrow k_\lambda}^{H^0(P_s/B,\, k_\lambda)}$$

Here $\leftarrow$ stands for the action of $Y_\alpha \in \mathfrak{g}_{-\alpha} \subset \mathfrak{b}$.

All other $Y_\beta \in \mathfrak{b}$ for $\beta \neq \alpha$ act trivially. $\square$

Thm A: Let $s \in S$, $\alpha = \alpha_s \in \Delta$ and $\lambda \in X(T)$, s.t.,

$\langle \lambda + \rho, \alpha^\vee \rangle \geq 0$. Then, there is an isomorphism of $G$-modules

$$H^n(G/B, \mathcal{L}_\lambda) = H^{n+1}(G/B, \mathcal{L}_{s.\lambda})$$

Proof: For simplicity assume that $G$ is simply-connected.

Then $\rho \in X(T)$. Assume that $\langle \lambda + \rho, \alpha^\vee \rangle \geq 2$.

Write $V = H^0(P_s/B, \mathcal{L}_{\lambda+\rho})$ and $W = H^0(P_s/B, \mathcal{L}_{\lambda-\rho-\alpha})$.

By the Proposition, there are s.e.s. of $\mathcal{O}_{G/B}$-modules

$$0 \to \underbrace{\mathcal{L}_K \otimes \mathcal{L}_{-\rho}}_{M} \to \mathcal{L}_V \otimes \mathcal{L}_\rho \to \mathcal{L}_\lambda \to 0 \quad \text{and}$$

$$0 \to \mathcal{L}_{s.\lambda} \to M \to V' \otimes \mathcal{L}_{-\rho} \to 0.$$

Since $\langle -\rho, \alpha^\vee \rangle = -1$ and $V$ and $V'$

are $P_s$- rep's, by "an easy" Lemma

$H^i(G/B, \mathcal{L}_V \otimes \mathcal{L}_{-\rho}) = H^i(G/B, \mathcal{L}_{V'} \otimes \mathcal{L}_{-\rho}) = 0 \ \forall i$

for all $n$. So, by the long exact sequences

for $H^n(\ )$, we get

$$0 \to H^n(G/B, \mathcal{L}_\lambda) \xrightarrow{\sim} H^{n+1}(G/B, M) \to 0 \quad \text{and}$$

$$0 \to H^{n+1}(G/B, \mathcal{L}_{s.\lambda}) \xrightarrow{\sim} H^{n+1}(G/B, M) \to 0 .$$

The case $\langle \lambda+\rho, \alpha^\vee \rangle = 0, 1$ is similar (simpler!) $\square$

Cor A: Let $\lambda \in X(T)$ with $\lambda+\rho$ dominant.

Then $H^n(G/B, \mathcal{L}_\lambda) \cong H^{n+\ell(w)}(G/B, \mathcal{L}_{w\cdot\lambda})$

for all $w \in W$

Proof: Induction on $\ell(w)$ and Thm A. $\square$

Cor B: If $\lambda \in X(T)$ is dominant, $H^n(G/B, \mathcal{L}_\lambda) = 0$

for all $n > 0$

Proof: $H^n(G/B, \mathcal{L}_\lambda) \cong H^{n+\ell(w_0)}(G/B, \mathcal{L}_{w_0\cdot\lambda}) = 0$

since $\ell(w_0) + n = \dim G/B + n > \dim G/B$ using

1.30 Facts $\square$

**Cor C:** Let $\lambda \in X(T)$ and $\alpha \in \Phi$, s.t.,
$\langle \lambda+\rho, \alpha^\vee \rangle = 0$. Then $H^n(G/B, \mathcal{L}_\lambda) = 0 \ \forall n$.

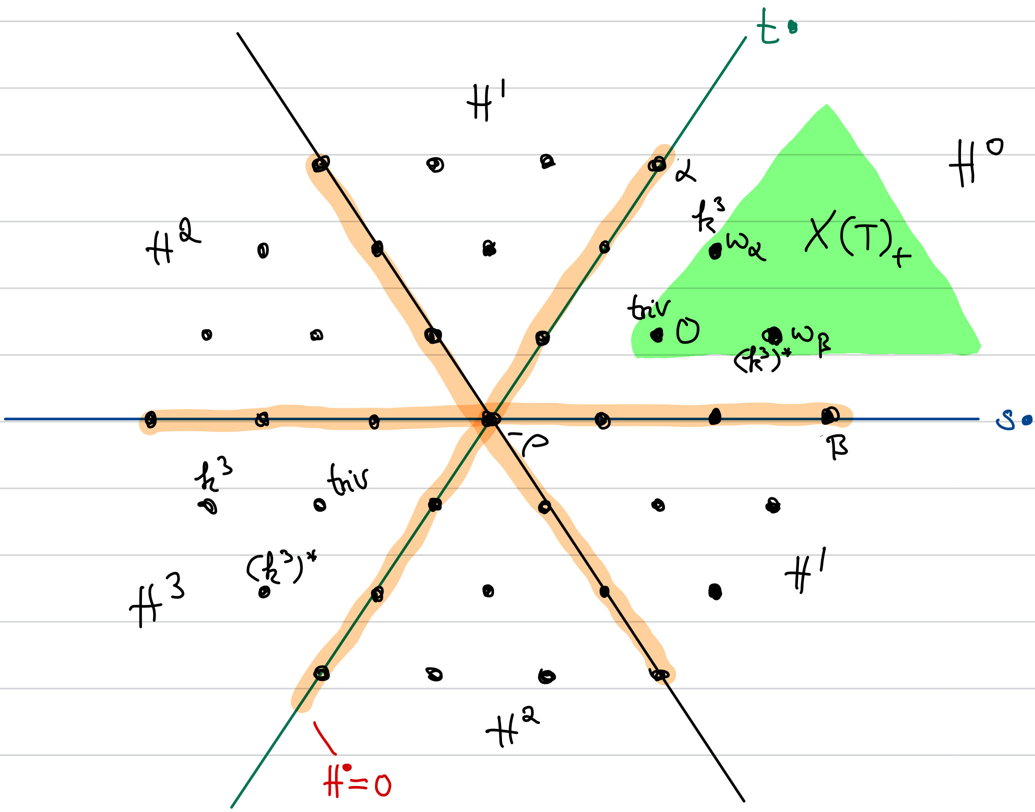
**Proof:** Let $w \in W$, s.t. $w(\alpha) \in \Delta$. Then
$\langle w \bullet \lambda, w(\alpha)^\vee \rangle = \langle w(\lambda+\rho) - \rho, w(\alpha)^\vee \rangle = -1$ and
$H^n(G/B, \mathcal{L}_\lambda) \cong H^{n+\ell(w)}(G/B, \mathcal{L}_{w\bullet\lambda}) = 0$
by the Lemma. $\square$

**Thm:** Borel-Weil-Bott. Let $\lambda \in X(T)$.
Then either, there is $\alpha \in \Phi$, s.t. $\langle \lambda+\rho, \alpha^\vee \rangle = 0$ and
$H^n(G/B, \mathcal{L}_\lambda) = 0 \ \forall n$, or there is a unique
$w \in W$, s.t. $w \bullet \lambda$ is dominant and

$$H^n(G/B, \mathcal{L}_\lambda) = \begin{cases} H^0(G/B, \mathcal{L}_{\omega\cdot\lambda}) = L(\omega\cdot\lambda) & \text{if } n = \ell(\omega) \\ 0 & \text{else.} \end{cases}$$

<u>Proof</u>: Use Cor A – C $\square$

Visualisation: $G = SL_3$ $S = \{s, t\}$, $\alpha = \alpha_s$, $\beta = \alpha_t$



## 1.34 Demazure operators

<u>Def:</u> For $s \in S$, define

$$\Delta_s : R_T \to R_T, \quad e^{\lambda} \mapsto \frac{e^{\lambda} - e^{s \cdot \lambda}}{1 - e^{-\alpha}}$$

Let $w \in W$ and $\underline{w} = (s_1, \dots, s_n)$ be a reduced expression. Then

$$\Delta_{\underline{w}} = \Delta_{s_1} \cdots \Delta_{s_n} : R_T \to R_T$$

is called the Demazure operator associated to $\underline{w}$ $\square$

<u>Lemma A</u> Let $s \in S$, then

$$\Delta_s(e^\lambda) = \sum_i (-1)^i \operatorname{ch}(H^i(P_s/B, \mathcal{L}^\lambda)) =: \underset{\substack{\uparrow \\ \text{Euler characteristic}}}{\chi_T}(P_s/B, \mathcal{L}^\lambda)$$

<u>Proof</u>: Write $\chi = \chi_T(P_s/B, \mathcal{L}_\lambda)$. There are three cases:

<u>$\langle \lambda, \alpha^\vee \rangle \geq 0$</u>: Then

$$\chi = e^{s(\lambda)} + \dots + e^\lambda = e^{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha} + \dots + e^{\lambda - \alpha} + e^\lambda$$

$$\Rightarrow \chi(1 - e^{-\alpha}) = e^\lambda - e^{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha - \alpha}$$

$$= e^\lambda - e^{s(\lambda) - \alpha} = e^\lambda - e^{s \cdot \lambda}. \quad //$$

<u>$\langle \lambda, \alpha^\vee \rangle = -1$</u>: Then $s \cdot \lambda = s(\lambda) - \alpha = \lambda + \alpha - \alpha = \lambda$

so $\Delta(e^\lambda) = 0$. On the other hand $\chi = 0$. //

$\underline{\langle \lambda, \alpha^\vee \rangle \leq -2}$: Then

$$\chi = -\left(e^{\lambda+\alpha} + \cdots + e^{s(\lambda)-\alpha}\right)$$

$$= -\left(e^{\lambda+\alpha} + e^{\lambda+2\alpha} + \cdots + e^{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha - \alpha}\right)$$

So $\chi(1-e^{-\alpha}) = e^{\lambda} - e^{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha - \alpha}$

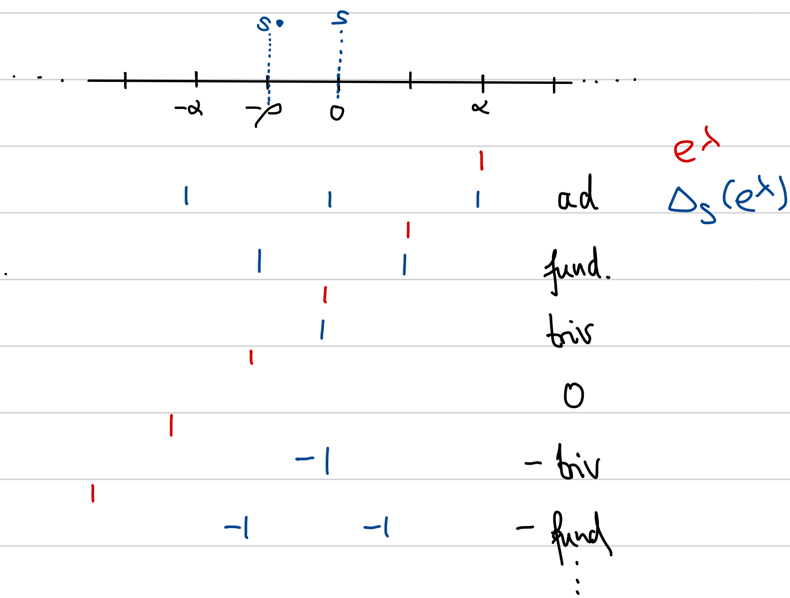
$$= e^{\lambda} - e^{s\cdot\lambda} \qquad \square$$

__Corollary A__: Let $V \in \operatorname{Rep} T$. Then

$$\Delta_s(\operatorname{ch}(V)) = \sum (-1)^i \operatorname{ch} H^i(P_s/B, \mathcal{L}_V) = \chi_T(P_s/B, \mathcal{L}_V)$$

__Proof__: Both sides are additive in $V$ $\square$

**Example**: Demazure operator for $SL_2$




$\square$

Recall that for $\underline{w} = (s_1, \ldots, s_n) \in S^n$

$$\pi_{\underline{w}}\ BS(\underline{w}) = P_{s_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_w}/B \to G/B.$$

For $V \in \operatorname{Rep} T$, we can define a bundle $\mathcal{L}_V$ on $BS(\underline{w})$, via $P_{s_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_w} \overset{B}{\times} V$.

There are natural maps

$$\pi : BS(\underline{w}s) \rightleftarrows BS(\underline{w}) : i$$

where $\pi$ is a $P_s/B = \mathbb{P}^1$-fibration with section $i$.

Cor B: Let $\underline{w} = (s_1, \ldots, s_n) \in S^n$ and $V \in \mathrm{Rep}\, T$. Then

$$\Delta_{\underline{w}}(\mathrm{ch} V) = \chi_T(BS(\underline{w}), \mathcal{L}_V)$$

Proof: Assume that the statement is true for $\underline{w}$ and let $s \in S$. Then, the Leray spectral for $\pi : BS(\underline{w}s) \to BS(\underline{w})$ degenerates and

$$H^n(BS(\underline{w}s), \mathcal{L}_V)) = \bigoplus_{i=0,1} H^{n+i}(BS(\underline{w}), H^i(P_s/B, \mathcal{L}_V))$$

So, we obtain

$$\chi_T(BS(\underline{w}s), \mathcal{L}_V) = \sum_{i=0,1} (-1)^i \chi(BS(\underline{w}), H^i(P_s/B, \mathcal{L}_V))$$

$$= \Delta_{\underline{w}} \Delta_s(\mathrm{ch} V) = \Delta_{\underline{w}s}(\mathrm{ch} V)$$

using Lemma A. $\square$

Remark: We hence obtain

$$\begin{tikzcd}
{[V]} \arrow[r, mapsto] & \sum (-1)^n [H^n(\mathcal{BS}(w), \mathcal{L}_V)] \\
K_0(\operatorname{Rep} T) \arrow[r] \arrow[d, "ch", "\wr"'] & K_0(\operatorname{Rep} T) \arrow[d, "ch", "\wr"'] \\
R_T \arrow[r, "\Delta_{\underline{w}}"] & R_T
\end{tikzcd}$$

$\square$

## 1.35 Demazure character formula

<u>Hard Fact:</u> Let $w \in W$ with reduced expression $\underline{w} = (s_1, \dots, s_n)$ and $V \in \operatorname{Rep} T$. Then

$$H^n(BS(\underline{w}), \mathcal{L}_V) \cong H^n(\overline{X}_w, \mathcal{L}_V) \qquad \square$$

– Andersen - Schubert varieties and Demazure's character formula

<u>Thm:</u> Let $w \in W$ and $\underline{w} = (s_1, \dots, s_n)$ a reduced expression. Then

$$\chi_T(\overline{X}_w, \mathcal{L}_V) = \Delta_{\underline{w}}(\operatorname{ch} V)$$

<u>Proof:</u> Hard fact + 1.34 Cor B $\square$

Cor A: $\Delta_w = \Delta_{\underline{w}}$ does not depend on the choice of reduced expression

Cor B: Let $\lambda \in X(T)_+$. Then

$$\operatorname{ch} L(\lambda) = \Delta_{w_0}(e^\lambda) \qquad \square$$

Lemma: let $s \in S$ and $\alpha = \alpha_s \in X(T)$. Then

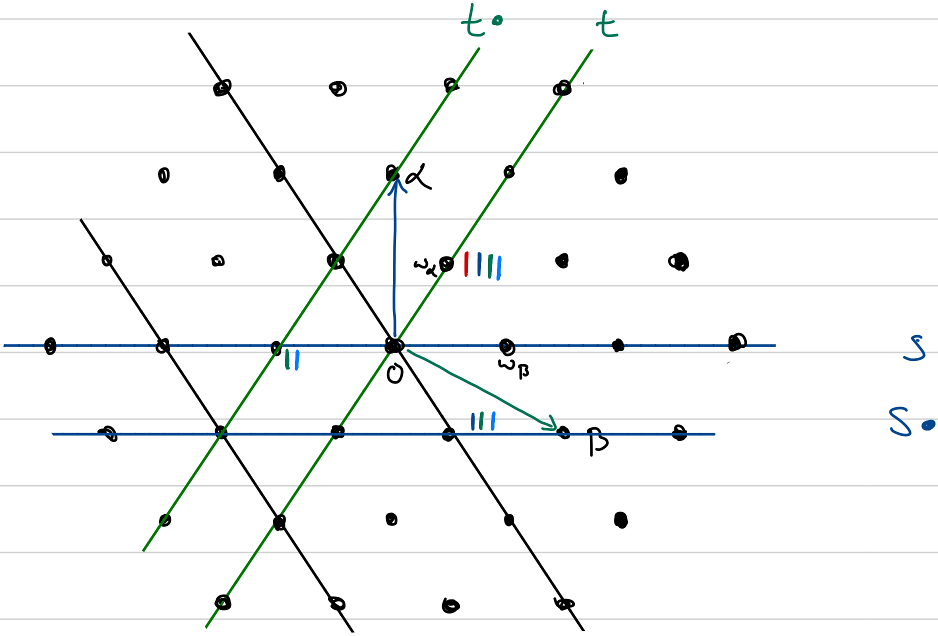
(1) $\Delta_s(1) = 1$ (2) $\Delta_s^2 = \Delta_s$ (3) $s\Delta_s = \Delta_s$

(4) $\Delta_s$ is $R_T^W = R_g$ linear

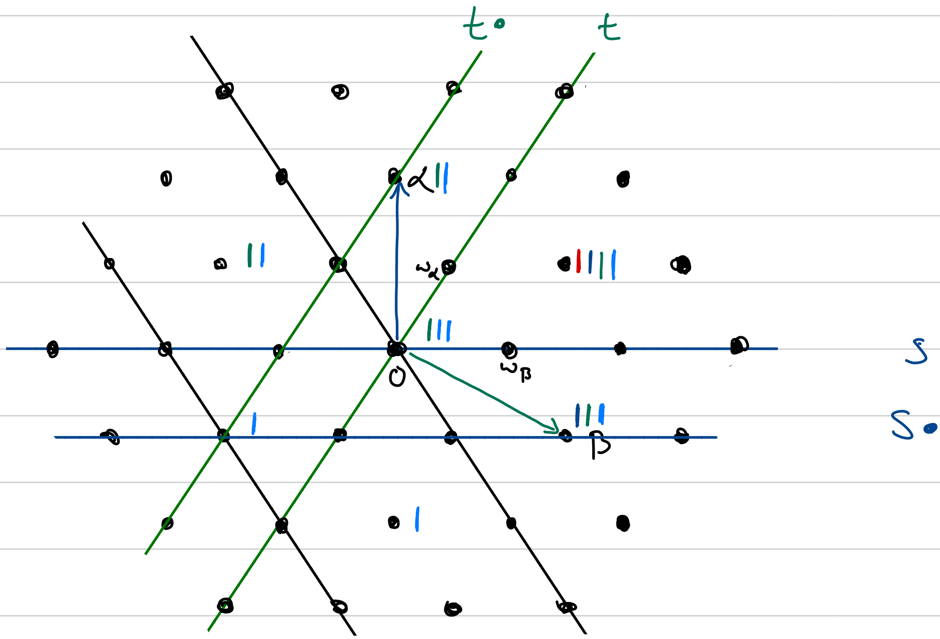
Example: $SL_3$, $S = \{s, t\}$, $\alpha = \alpha_s$, $\beta = \alpha_t$

$$\Delta_s \Delta_t \Delta_s e^{\lambda} = \mathrm{ch}\, L(\lambda)$$

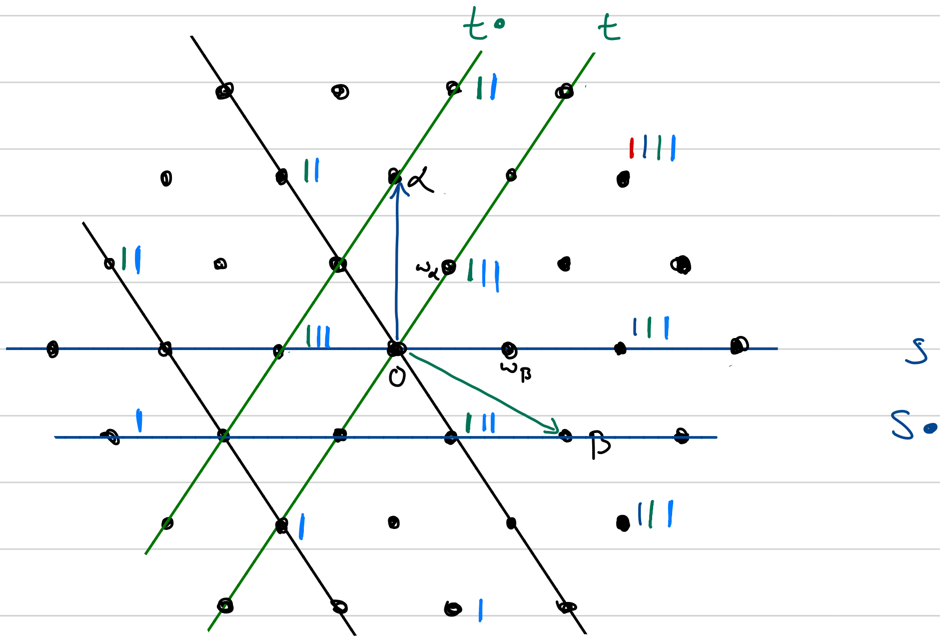
$\lambda = \omega_\alpha$, $L(\lambda) = k^3 =$ fund.



$\lambda = \rho \;,\; L(\lambda) = ad$



$\lambda = \rho + \omega_\alpha$



# 2. Equivariant K-theory and Weyl's character formula

## 2.1 Equivariant Sheaves

Let $G$ be a linear algebraic group and $X$ a variety with $G$-action (Short: $X$ is a $G$-variety)

Consider the action and projection maps:

$$a, p : G \times X \rightrightarrows X$$

<u>Def</u> A: A $G$-equivariant coherent sheaf is a coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X)$ and an isomorphism

$$\varphi : a^* \mathcal{F} \xrightarrow{\sim} p^* \mathcal{F}$$

such that the following cocycle condition holds

$$\pi_1^*(\varphi) \circ \pi_2^*(\varphi) = \pi_3^*(\varphi)$$

for

$$\mathcal{G} \times \mathcal{G} \times X \overset{\pi_1}{\underset{\pi_3}{\overset{\xrightarrow{\pi_2}}{\rightrightarrows}}} \mathcal{G} \times X$$

$$\pi_i(g,h,x) = \begin{cases} (h,x) & i=1 \\ (g,hx) & i=2 \\ (gh,x) & i=3 \end{cases}$$

Denote the category of $\mathcal{G}$-equivariant coh. sheaves by

$\mathrm{Coh}_{\mathcal{G}}(X)$ $\square$

<u>Def B</u>: A $\mathcal{G}$-equivariant vector bundle on $X$ is a vector bundle $\pi: E \to X$ with a linear action of $\mathcal{G}$ on $E$ commuting with $\pi$.

Denote the category of $G$-equivariant vector bundles by $\mathrm{Vec}_G(X)$ $\square$

<u>Remark</u>: (1) Passing to the sheaf of sections yields an equivalence between $G$-equivariant vector bundles and locally free coherent sheaves

$$\begin{tikzcd} \mathrm{Vec}_G(X) \arrow[r, hook] \arrow[dr, "\sim"'] & \mathrm{Coh}_G(X) \\ & \{\text{locally free}\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

We treat $\mathrm{Vec}_G(X) \subset \mathrm{Coh}_G(X)$ as a subcategory

(2) $\mathcal{O}_X$ is naturally a $G$-equivariant (Why?)

(3) For $X = pt$, $\mathrm{Coh}_G(pt) = \mathrm{Rep}(G)$

(4) $\mathrm{Coh}_{\mathcal{G}}(X)$ and $\mathrm{Vec}_{\mathcal{G}}(Y)$ inherit the usual functors $f_*, f^*$ for $\mathcal{G}$-equivariant maps $f: X \to Y$ $\square$

__Thm A:__ If $X$ is a smooth quasi-proj. $G$-variety.

Then $\mathcal{F} \in Coh_G(X)$ admits a resolution

$$0 \to \mathcal{F}^{\dim(X)} \to \quad \cdots \to \mathcal{F}^1 \to \mathcal{F} \to 0$$

with $\mathcal{F}^i \in Vec_G(X)$ $\square$

__Thm B:__ Let $H \subset G$ and $X$ an $H$-variety. Then there is an equivalence

$$Coh_H(X) \simeq Coh_G(G \overset{H}{\times} X) \qquad \square$$

__Examples:__ In Thm B, let $X = pt$. Then we obtain an equivalence

$$Rep(H) \to Coh_G(G/H)$$

which maps $V$ to $\mathcal{L}_V$. In particular, we can interpret induction as the composition

$$\begin{tikzcd} \mathrm{Rep}(H) \arrow[rr, "\mathrm{Ind}_H^G"] \arrow[d, "\wr"] & & \mathrm{Rep}(G) \arrow[d, "\wr"] \\ \mathrm{Coh}_H(pt) \arrow[r, "\sim"] & \mathrm{Coh}_G(G/H) \arrow[r, "\pi_*"] & \mathrm{Coh}_G(pt) \end{tikzcd}$$

$\square$

## 2.2. Equivariant K-theory.

Recall that for an Abelian category, the Grothendieck group is defined by

$$K_0(\mathcal{A}) = \mathbb{Z}\,\mathrm{Ob}(\mathcal{A}) \Big/ \left\{ \begin{array}{l} [A]+[C] = [C] \text{ for} \\ 0 \to A \to B \to C \to 0 \end{array} \right\}$$

The same definition works for subcategories of Abelian categories ($\rightsquigarrow$ exact categories)

<u>Def</u>: Let $X$ be a $G$-variety. Then

$$K_G(X) = K_0(\mathrm{Vec}_G(X))$$

is called the <u>$G$-equivariant K-theory</u> of $X$.

Thm: Let $X$ be a smooth quasiprojective $G$-variety. Then the natural map

$$K_0(\mathrm{Vec}_G(X)) \to K_0(\mathrm{Coh}_G(X))$$

is an isomorphism $\square$

Proof: We only show surjectivity:

$$[\mathcal{F}] = \sum_{i=1}^{\dim X} (-1)^i [\mathcal{F}_i]$$

for $\mathcal{F} \in \mathrm{Coh}_G(X)$ and $\cdots \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}$ a resolution with $\mathcal{F}_i \in \mathrm{Vec}_G(X)$ as in 2.1. Thm A $\square$

Remark: $K_0(\mathrm{Coh}_G(X))$ is the $G$-equivariant $G$-theory
$\phantom{x}$ || Grothendieck

For $X$ singular, $K$- and $G$-theory can differ.

There is a helpful analogy:

K-th. is to G-th.

as

cohomology is to Borel-Moore homology.

So the Thm. can be seen as a "Poincare duality" between K-th. and G-th. $\square$

## 2.3. Functorialities for K-theory

Let $f: X \to Y$ be a map of $G$-varieties.

<u>Pullback:</u>

Then pullback defines an exact functor

$$f^*: \mathrm{Vec}_G(Y) \longrightarrow \mathrm{Vec}_G(X) \quad \text{via}$$

$$f^*(E) = \begin{tikzcd} E \times_Y X \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

We hence obtain a pullback in equivariant K-theory

$$f^*: K_G(Y) \longrightarrow K_G(Y), \quad [E] \mapsto [f^*(G)]$$

<u>Pushforward</u>: Assume that $f$ is proper. Then the pushforward $f_*: \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$ yields a a left exact functor:

$$f_*: Coh_G(X) \to Coh_G(Y)$$

This yields a map:

$$f_*: K_0(Coh_G(X)) \longrightarrow K_0(Coh_G(Y))$$

$$[\mathcal{F}] \longmapsto \sum (-1)^i [R^i f_* \mathcal{F}]$$

If $X, Y$ are both smooth quasi-projective, we obtain a pushforward

$$f_*: K_G(X) \to K_G(Y)$$

Tensor product: The tensor product of vector bundles yields a multiplication

- $K_G(X) \otimes K_G(X) \to K_G(X)$, $[E] \otimes [F] \mapsto [E \otimes F]$.

This turns $K_G(X)$ into a ring. $f^*$ is a ring homomorphism.

Projection Formula: If $f$ is proper, the projection formula

$$R^n f_* \mathcal{F} \otimes f^* E = (R^n f_* \mathcal{F}) \otimes E$$

(see 1.31) implies that in eq. K-th.

$$f_*(- \otimes f^*(-)) = (f_* -) \otimes - \quad .$$

**Flat base change** Consider a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "g"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y' \end{tikzcd}$$

such that $f$ is proper and $g$ flat. Then

$$f_* g^* = g^* f_*$$

on eq. K-theory.

## 2.4 Idea of Weyl character formula:

Let $X = G/B$ be the flag variety.

Then we obtain the diagram:

$$\begin{tikzcd} wB/B \in X^T \arrow[r, hook, "i"] \arrow[d, equal, "\leftarrow \text{identify}"] & X \arrow[r, "\pi"] & pt \\ w \in W & & \end{tikzcd}$$

which induces on equivariant K-theory:

$$\begin{tikzcd} K_T(X^T) \arrow[r, shift left, "i_*"] \arrow[d, equal] & K_T(X) \arrow[l, shift left, "i^*"] \arrow[r, "\pi_*"] & K_T(pt) \arrow[d, equal] \\ \bigoplus_{w \in W} R_T \arrow[rr, dashed, "\Sigma"] & & R_T \end{tikzcd}$$

It is easy to compute $\pi_* i_* i^* \mathcal{L}_\lambda = \sum_{w \in W} e^{w(\lambda)}$.

But, $i_*$ and $i^*$ are not inverse and $i_* i^* \mathcal{L}_\lambda \neq \mathcal{L}_\lambda$.

So we need a "correction term". To find this, it is easier to study the composition $i^* i_*$ first.

For $w \in W$, denote the inclusion by $i_w$

Then we find a $T$-invariant "tubular neighborhood"

$$\begin{tikzcd} \{w\} \arrow[r, hook, "s_w"] \arrow[rr, bend left, "i_w"] \arrow[rrr, bend right, "="] & N_w \arrow[r, "j_w"] & X \arrow[r, "\pi"] & pt \end{tikzcd}$$

s.t. $N_w \cong T_w X \leftarrow$ tangent space TODO 1

$$i^* i_* = \sum i_w^* i_{w,*} = \sum s_w^* j_w^* j_{w*} s_{w*}$$

$$= \sum s_w^* s_{w*} \underset{\text{TODO 2}}{=} e$$

Then, we get $(e^{-1} i^*) i_* = \mathrm{id}$

Assume that also $i_*(e^{-1} i^*) = \mathrm{id}$ TODO3

Then $\pi_* \mathcal{L}_\lambda = \pi_* \underbrace{i_* e^{-1} i^* \mathcal{L}_\lambda}_{\text{"easy" to compute}}$ TODO4

$\Downarrow$

Weyl character formula.

## 2.5 Koszul Complex and Self-Intersection Formula

We tackle TODO 2

Example: (1) Consider the T-equivariant maps

$$k_\lambda \underset{s}{\overset{p}{\rightleftarrows}} \{0\}$$

Since $k_\lambda$ is affine we work with $\mathcal{O}(k_\lambda) = \mathrm{Sym}(k_\lambda^*) = k[x]$ modules. Let $k_\mu \in \mathrm{Vec}_T(\{0\}) = \mathrm{Rep}(T)$, the trivial rep.

Then $s_* k_\mu$ is a skyscraper sheaf at $\{0\}$ with resolution in $\mathrm{Vec}(k_\lambda)$

$$k[x] \xrightarrow{\cdot x} k[x] \to s_* k_\mu$$

To make the resolution T-equivariant, we need to take into account that $T$ acts on $x^n$ via

$t \cdot x^n = \lambda^{-n}(t) \cdot x^n$ and on $s_* k_\mu$ via $\mu(t)$.

A $T$-equivariant resolution is hence given by

$$
\begin{array}{ccccc}
k_\lambda^* \otimes k_\mu \otimes k[x] & \longrightarrow & k_\mu \otimes k[x] & \longrightarrow & s_* k_\mu \\
 & & \| & & \| \\
p^*(k_\lambda^* \otimes k_\mu) & \longrightarrow & p^* k_\mu & \longrightarrow & s_* k_\mu
\end{array}
$$

Hence, in $K_T(k_x)$ we obtain that

$$s_* e^\mu = p^* e^\mu - p^* e^\lambda e^\mu$$

So that $\quad s_*(-) = p^*((1-e^\lambda) - )$

(2) To generalize this, let $V \in \mathrm{Rep}\,T$, $n = \dim V$ and

$$p : V \rightleftarrows \{0\} : s$$

Then $s_* W$ has a <u>Koszul-resolution</u>:

$$p^*(\Lambda^n V^* \otimes W) \to \dots \to p^*(\Lambda^1 V^* \otimes W) \to p^*(\Lambda^0 V^* \otimes W) \to s_* W$$

so that

$$s_*(-) = p^*\Big(\underbrace{\sum_{i=0}^{n} (-1)^i \Lambda^i V^*}_{= \lambda(V^*)} \otimes W\Big)$$

In particular, since $s^* p^* = \mathrm{id}$

$$s^* s_* = \lambda(V^*) \cdot -$$

**Def**: Let $X$ be a $T$-variety and $V \in Vect_T(X)$.

Then define $\lambda(V) = \sum (-1)^i [\Lambda^i V] \in K_T(x)$.

**Thm A**: Let $p: V \rightleftarrows X : s$

be a vector bundle with zero section.

Then $s^* s_* = \lambda(V^*) \cdot -$ on $K_G(X)$.

**Proof Sketch**: Trivialize, reduce to $X = pt$ then use

Example (2).

**Thm B** (Self-intersection formula) let $i: Z \hookrightarrow X$ be a

closed $T$-equivariant embedding with $Y, X$ smooth.

Then $i^* i_* = \lambda(T_Z^* X) \cdot -$

<u>Proof Sketch</u>: Deformation to the normal cone:

The pair $(X, Z)$ can be deformed to $(T_Z X, Z)$ over $\mathbb{A}^1$, K-theory is compatible with deformation. Use Thm B. $\square$

<u>Thm C</u>: (Thom isomorphism) let $p : V \rightleftarrows X : s$ be a vector bundle with zero section. Then

$$p^* : K_g(X) \to K_g(V) : s^*$$

are inverse isomorphisms

<u>Proof</u> Omitted $\square$

## 2.6 Atiyah-Bott Localization in K-Theory.

We tackle TODO3. Let $X$ be a smooth projective $T$-variety such that $X^T \overset{i}{\hookrightarrow} X$ is discrete.

Choose a regular cocharacter $\eta : \mathbb{G}_m \to T$, so that $X^{\eta(\mathbb{G}_m)} = X^T$. For $w \in X^T$, denote by

$$X_w^+ = \{ x \in X \mid \lim_{t \to 0} \eta(t) x = w \}$$

the attracting cell. Then by a Theorem of Bialynicki-Birula, $X = \biguplus_{w \in W} X_w^+$ is a stratification of $X$, and $X_w^+ \cong (T_w X)_{>0} \subset T_w X$ is an affine space.

Using the long exact sequence in K-theory and the Thom-Isom., one may show that (non-canonically)

$$K_T(X) \cong \bigoplus_{w \in X^T} K_T(X_w^+) \cong \bigoplus_{w \in R^T} R_T$$

is a free rank $|X^T|$ $R_T$-module.

We denote $Q = \mathrm{Quot}(R_T)$ and

$M_Q = M \otimes_{R_T} Q$ for an $R_T$-module.

Thm (Atiyah-Bott for K-theory) For $i = X^T \hookrightarrow X$, consider

$$i_* : K_T(X^T) \rightleftarrows K_T(X) : i^*$$

$$K_T(X^T) = \bigoplus_{w \in X^T} R_T \ni e = (\lambda(T_w^* X))_w$$

Then

(a) $i^* i_* = e \cdot -$

(b) $i_* : K_T(X^T)_Q \rightleftarrows K_T(X)_Q : e^{-1} i^*(-)$

are mutually inverse isomorphisms.

Pf: (a) is the self-intersection formula 2.5. Thm B.

(b) $i_*$ is injective and both sides have $\dim = |X^T|$ over $Q$.

## 2.7 The Weyl character formula

We now tackle TODO1 and TODO4.

Again let $W \cong X^T \overset{i}{\hookrightarrow} X = G/B$.

Recall that $T_e X = T_{B/B} G/B = \mathfrak{g}/\mathfrak{b}$. Similary

$T_w X = w(\mathfrak{g}/\mathfrak{b})$. As $T$-module $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^-$.

By our choice of roots, $\mathfrak{n}^-$ has weight $\Phi^+$.

<u>Lemma</u> $\lambda(T_w^* X) = (-1)^{\ell(w)} e^{-w \cdot 0} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \in R_T$

<u>Proof</u> $\lambda(T_w^* X) = \lambda(w(\mathfrak{n}^-)) = \lambda\left(\bigoplus_{\alpha \in \Phi^+} k^*_{w(\alpha)}\right)$

$= \lambda\left(\bigoplus_{\alpha \in \Phi^+} k_{-w(\alpha)}\right) = \prod_{\alpha \in \Phi^+} \lambda(k_{-w(\alpha)}) = \prod_{\alpha \in \Phi^+} \left(1 - e^{-w(\alpha)}\right)$

<u>Exercise</u>

Let $\Delta = \prod_{\alpha \in \Phi} (e^{\alpha/2} - e^{-\alpha/2})$. Then for $s \in S$,

$$s(\Delta) = -\Delta \qquad \text{(Exercise)}$$

$$\Rightarrow \quad w(\Delta) = (-1)^{\ell(w)} \Delta .$$

Moreover $e^{-\rho}\Delta = \prod (1 - e^{-\alpha})$.

Hence $w(\Delta e^{-\rho}) = (-1)^{\ell(w)} e^{w(-\rho)} \Delta$

$$= (-1)^{\ell(w)} e^{-w \bullet 0} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \quad \square$$

Thm: (Weyl character formula) Let $\lambda \in X(T)$.

$$\sum_i (-1)^i \operatorname{ch} H^i(G/B, \mathcal{L}_\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w\cdot\lambda}}{\prod_{\alpha\in\Phi^+} (1-e^{-\alpha})}$$

Proof: Let $\pi: X = G/B \to pt$. Then the LHS

is obtained as $\pi_*[\mathcal{L}_\lambda] \in K_T(pt) = R_T$.

Let $e = (\lambda(T_w X))_{w\in W}$. Moreover $i^*\mathcal{L}_\lambda = (e^{w(\lambda)})_{w\in W}$

By Atiyah-Bott for K-theory,

$$\pi_*[\mathcal{L}_\lambda] = \pi_* i_* e\, i^*[\mathcal{L}_\lambda] = \sum_{w\in W} \lambda(T_w X)\, e^{w(\lambda)}$$

$$= \sum_{w\in W} \frac{e^{w(\lambda)}}{(-1)^{\ell(w)} e^{w\cdot 0} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})}$$

The statement follows using $w(\lambda) + w\cdot 0 = w\cdot\lambda$ $\square$

# 4 Summary

## Algebraic groups

$\mathcal{G}$ affine algebraic group $\Leftrightarrow$ $\mathcal{O}(\mathcal{G})$ comm. red. f.g. Hopf algebra

$\Updownarrow$

$\mathcal{G} \subset \mathcal{GL}_n$ linear algebraic group

$\mathcal{G} \rightsquigarrow \mathcal{O}(\mathcal{G})$

# Quotients

$$H \subset G \quad \rightsquigarrow \quad \begin{array}{ccc} I(H) & \subset & \mathcal{O}(G) \\ \cup & & \cup \\ U & \subset & W \quad \text{f.d.} \\ & \downarrow & \\ L = \Lambda^d U & \subset & \Lambda^d W \\ & \downarrow & \\ x = [L] & \subset & \mathbb{P}(V) \\ & \downarrow & \\ G/H & \cong & Gx \end{array} \qquad \begin{array}{c} H = G_{I(H)} \\ \\ H = G_U \\ \\ H = G_L \\ \\ H = G_x \\ \\ \text{quasi projective} \end{array}$$

## Lie algebras

$$G \rightsquigarrow \mathfrak{g} = \mathrm{Lie}(G) = T_e X = \mathrm{Der}_g(\mathcal{O}(G))$$

$$GL_n \rightsquigarrow \mathfrak{gl}_n$$

$$\mathrm{Ad}: G \to \mathfrak{g}, \quad \mathrm{ad}: \mathfrak{g} \to \mathfrak{g}$$

$$\text{"} gYg^{-1} \text{"} \qquad \text{"} [X,Y] \text{"}$$

## Anatomy of a group element

$$X = X_s + X_n, \qquad x = x_s x_u = x_u x_s$$

$\rightsquigarrow$ Intrinsical!

# Anatomy of a group

$$\{1\} \;—\; R_u(G^\circ) \;—\; R(G^\circ) \;—\; G^\circ \;—\; G$$

$$\underbrace{\{1\} \;—\; R_u(G^\circ) \;—\; R(G^\circ) \;—\; G^\circ}_{\text{connected}} \;—\; \underbrace{G^\circ \;—\; G}_{\text{finite}}$$

$$\underbrace{\{1\} \;—\; R_u(G^\circ) \;—\; R(G^\circ)}_{\text{solvable}} \;—\; \underbrace{R(G^\circ) \;—\; G^\circ}_{\text{semisimple}}$$

solvable $\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$ semidirect product

semisimple = almost prod. of almost simple $\rightsquigarrow SL_n, PGL_n, \dots$

unipotent $\rightsquigarrow G_a, \begin{pmatrix} 1 & * \\ & \ddots & \\ & & 1 \end{pmatrix}$

torus $\rightsquigarrow G_m^n$

reductive $\rightsquigarrow GL_n$

# Diagonalizable Groups

$H \subset \mathbb{G}_m^n \iff H = H_{ss}$

$X(H)$, $Y(H)$, $\langle , \rangle$

$$\{\text{diag. groups}\} \overset{\sim}{\longleftrightarrow} \{\text{f.g. Abelian groups}\}^{op}$$

$$H \longmapsto X(H)$$

$$\mathrm{Spec}(k[M]) \longleftarrow M$$

# Flag variety

torus Borel reductive

$$T \subset \underbrace{B \subset G}$$

$$G/B \text{ projective}$$

$$\cong$$

$$\mathcal{B} = \{\text{Borel subgroups}\} \cong \{\text{Borel subalgebras}\}$$

$$\|$$

"flag variety"

$$W = W(G,T) \curvearrowright \mathcal{B}^T$$

$$\uparrow$$

torsor

$G$ s.s. rank 1

$$\begin{tikzcd}
SL_2 \arrow[r] \arrow[dr] & G \arrow[r] \arrow[d] & PGL_2 = \mathrm{Aut}(PGL_2) \arrow[dl] \\
 & \mathcal{B} = \mathbb{P}^1 &
\end{tikzcd}$$

## Weights and Roots

$$\Phi_{red} \underset{\text{If } \mathcal{G} \text{ reductive}}{\overset{}{\subset}} \Phi \subset X(T)$$

If $\mathcal{G}$ reductive $=$

$\mathrm{Lie}(\mathcal{G}/R(\mathcal{G}))$

$$R(\mathcal{G}) = \bigcap_{B \in \mathcal{B}} B$$

## Root Subgroups

$\mathcal{G}$ reductive $\quad \alpha \in X(T)$

$\rightsquigarrow T_\alpha \subset Z_\alpha \leftarrow SL_2$

$$U_\alpha \longleftarrow \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$

$$T \supset T_\alpha \longleftarrow \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \leftarrow\!\shortmid t \quad \alpha^\vee$$

coroot

## Root Datum

$(X(T), \Phi, Y(T), \Phi^\vee) \iff T \subset G$

root datum $\qquad\qquad$ reductive group

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

$(X(T)_{\mathbb{R}}, \Phi) \qquad\qquad \mathfrak{t} \subset \mathfrak{g}$

root system $\qquad\qquad$ Lie alg.

$\|$

$A\ B\ C \cdots H$

## W-torsors

$T \subset B \subset G$

$\Phi^+ \subset \Phi$

$\Delta \subset \Phi$

$S \subset W$

## Tits System

$\mathcal{G}$ reductive

$$\Rightarrow (\mathcal{G}, B, N, S) \text{ Tits system}$$

$$\mathcal{G} = BWB, \quad \{B \subset P_I \subset \mathcal{G}\} \Leftrightarrow \{I \subset S\}$$

## Bruhat cells

$$U_w \times \{w\} \times T \times U \xrightarrow{\sim} BwB$$

$$\|$$

$$wU^-w^{-1} \cap U$$

$$\|$$

$$\prod_{\underbrace{w(\Phi^-)\cap\Phi^+}_{\# = \ell(w)}} U_\alpha$$

$$BwB/B \subset \mathcal{G}/B$$

$$\|$$

$$\mathbb{A}^{\ell(w)}$$

## Bott-Samelson.

$w = s_1 \cdots s_n$

$$\underset{\text{resolution}}{P_{s_1} \overset{B}{\times} \cdots \overset{B}{\times} P_{s_n} / B} \longrightarrow \underset{\text{Schubert cell}}{\overline{BwB/B}}$$